THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

ANALYSIS OF THE WEALTH DISTRIBUTION AT EQUILIBRIUM IN A HETEROGENEOUS AGENT ECONOMY

By

MUFFASIR BADSHAH

A Dissertation submitted to the
Department of Statistics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Degree Awarded:
Summer Semester, 2010

The members of the committee approve the dissertation of Muffasir Badshah defended on April 28, 2010.

Anuj Srivastava
Professor Co-Directing Dissertation

Paul Beaumont
Professor Co-Directing Dissertation

Wei Wu
Committee Member

Alec Kercheval
Outside Committee Member

Approved:

Dan McGee, Chair, Department of Statistics

Joseph Travis, Dean, College of Arts and Sciences

The Graduate School has verified and approved the above-named committee members.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

This paper aims at analyzing a macroeconomy with a continuum of infinitely-lived households that make rational decisions about consumption and wealth savings in the face of employment and aggregate productivity shocks. The heterogeneous population structure arises when households differ in wealth and employment status against which they cannot insure. In this framework, the household wealth evolution is modeled as a mixture Markov process. The stationary wealth distributions are obtained using eigen structures of transition matrices under the Perron-Frobenius theorem. This step is utilized repeatedly to find the equilibrium state of the system, and it leads to an efficient framework for studying the dynamic general equilibrium. A systematic evaluation of the equilibrium state under different initial conditions is further presented and analyzed.

CHAPTER 1

INTRODUCTION

In 1928 a young mathematician named Frank Ramsey proposed a dynamic model to answer a simple yet difficult question: "How much of its income should a nation save?" Ramsey's model lays a foundation for the macroeconomic theory, and variants of his dynamic optimization problem are the cornerstones of most models of economic fluctuations and growth. This paper reviews recent research aimed at solving a theoretical model of the macroeconomy (economy in a broad sense) with five key elements (i) it is based on rational decision-making by consumers and a single firm owned collectively by these consumers; (ii) it is dynamic, so that consumption and savings decisions are determined by intertemporal decisions (current decisions that take into account the future choices); (iii) it has stochastic aggregate shocks (random uncertainty at the economic level) which lead to upswings and downswings at a macroeconomic level; (iv) it considers general equilibrium, so that interest rates and wage rates are determined endogenously - determined by the interaction of entities in the given economy; and (v) it has a heterogeneous population structure where consumers differ in wealth and employment status against which they cannot insure.

Heterogeneous-agent based economies advance us a step closer to the study of 'real' economies, but they are harder to solve than homogeneous agent based economies where there is a single representative agent (or consumer) in the entire economy. In a heterogeneous-agent based economy, wealth is unevenly distributed among consumers, and part of the model solution is to determine the asymptotic wealth distribution. We call this the stationary wealth distribution and finding it will be the prime focus of our work. Furthermore, the stationary wealth distribution can be achieved at different initial conditions. Our focus is to find the stationary wealth distribution at equilibrium. The wealth distribution at stationary equilibrium helps economists answer questions such as "What would be the long term effects of changing wage rate (or unemployment benefits) on individual wealth?" Two kinds - competitive and stationary - of equilibrium are discussed in this paper. A competitive equilibrium is achieved using a time iterative technique while a stationary equilibrium is achieved using a fixed point method. A bridging explanation will be made to connect these two in chapter 4.

Algorithms to solve heterogenous agent models with endogenous wealth distributions have been introduced in economic literature in the past 15 years with notable studies in [1], [2], [3], [5], [6], [7], [9] and [12]. Most of these are iterative algorithms and take extremely long to converge if a solution exists at all. We use dynamic programming to solve the household optimization problem. It should be noted that the translation of the household optimization problem to a dynamic programming problem is not an easy task for non-elementary economies. Miao [15] discusses this issue and lays a theoretical foundation for problems in which dynamic programming can be used. We work with simple economies to exemplify our technique. The other problem arises when a continuous function (infinite-dimensional object) becomes a part of the consumer's state space. Ideally we would like to find an equilibrium for a continuum of agents with density as the state space argument. As one could imagine, estimating a continuous function accurately in a reasonable time frame in every period is no easy task. A variety of algorithms propose using projection and perturbation techniques to resolve running time issues. Some of these are reviewed in the next chapter. In this paper, we propose using eigen-analysis to compute intermediate and equilibrium stationary probability densities using the Perron-Frobenius Theorem. The approach is similar to [12] which maps the target eigenvector to a point in the probability function space using a cost function. At the stationary equilibrium, we look[1] for the limiting point in the aggregate parameter space. To find this point, we set a grid on the aggregate capital and search for a fixed point mapping in the Euclidean space. We also demonstrate this mapping using a binary search algorithm that is found to be much faster and more accurate. These mappings are highly nonlinear and involve a two tier process - optimization using dynamic programming to find the household (consumer) consumption and wealth policy, and computing the stationary density of the household wealth. Our approach is innovative and robust, and we use it to explain convergence failures reported by [4] in a model with idiosyncratic uncertainty. Once we formulate our method for the idiosyncratic case, we apply it to an economy with aggregate and idiosyncratic shocks (referred to as the general model). This general model is of prime interest to us. Our contributions are summarized as follows:

1. One problem with solving heterogeneous agent based economies is that the individual's state space has a continuous function (infinite-dimensional object). Estimating this density is challenging and leads to inaccuracies and longer running times. We propose an algorithm that avoids existing projection and perturbation techniques that lead to inaccuracies. We propose using eigen-analysis to compute intermediate and equilibrium stationary probability densities using the Perron-Frobenius Theorem.

2. The overall technique used here is a fixed point technique in contrast to the time iterative technique. We bridge these techniques for the case of an economy with no aggregate but with idiosyncratic shock.

3. Since fixed-point techniques take longer to converge relative to time iterative techniques, we propose a modification of our algorithm inspired by binary search that is faster and more accurate.

4. We set an aggregate capital grid and use our algorithm to explain convergence failures reported by [4] in a heterogeneous economy with idiosyncratic shock and no aggregate shock.

[1] [10] proves existence and uniqueness of the aggregate fixed points for the models considered in this paper.

5. For the model with both aggregate and idiosyncratic shocks, we investigate the existence or non-existence of the Stationary Equilibrium. The concern arises as agent's state space has an argument of cross-sectional wealth density that changes in every period.

6. On finding the stationary equilibrium in the economy with both idiosyncratic and aggregate shocks, we use our algorithms to estimate factor prices at the stationary equilibrium.

To describe the general model mentioned above, we need a mathematical object that is well suited to counting. This object is called measure. The key properties of measures are associated with the fact that they act as counting or weighting mechanisms. We review some measure theoretic concepts briefly below.

1.1 Mathematical Preliminaries

In this section, definitions are kept to a minimum (i.e. completion of measure, outer measure, etc have been omitted). In addition, relevant theorems, propositions and definitions are presented without proof.

Definition 1. *A class $\mathcal{A}$ (nonvoid) of subsets of A of a nonvoid set Ω is called a $\sigma-$field or a $\sigma-$algebra if:*

- *If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$*
- *If $A_1, A_2, ...$ is a countable collection of sets in $\mathcal{A}$, then the $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.*

Borel (defined below) $\sigma-$algebras are the $\sigma-$algebras generated by a family of open sets. If $\mathcal{A}$ is closed under complements and finite unions, then it is called a field or an algebra.

Proposition 1. *Closure under intersections*

- *Arbitrary intersections of fields (or $\sigma-$fields) are fields (or $\sigma-$fields).*
- *There exists a minimal field or $\sigma-$field generated by (or containing) any specified class $\mathcal{C}$ of subsets of Ω. For example,*

$$\sigma[\mathcal{C}] \equiv \cap \mathcal{F}_\alpha : \mathcal{F}_\alpha \textit{is a } \sigma\textit{-field of subsets of } \Omega \textit{ for which } \mathcal{C} \subset \mathcal{F}_\alpha$$

Definition 2. $(\Omega, \mathcal{A})$ *is called a measurable space if $\mathcal{A}$ is a $\sigma-$field of subsets of Ω*

Definition 3. *Consider a set function $\mu : \mathcal{A} \to [0, \infty]$ (that is $\mu(A) \geq 0$ for each $A \in \mathcal{A}$) having $\mu(\emptyset) = 0$.*

- *If $\mathcal{A}$ is a $\sigma-$field and μ is countable additive - that is, the measure of disjoint sets should be the sum of the measure of each - i.e.*

$$\mu\left(\sum_{n=1}^{\infty} A_n\right) = \left(\sum_{n=1}^{\infty} \mu(A_n)\right),$$

then μ is called a measure on $(\Omega, \mathcal{A})$. The triple $(\Omega, \mathcal{A}, \mu)$ is called a measure space. We call μ finite if $\mu(\Omega) < \infty$. We call μ $\sigma-$finite if there is a measurable decomposition of Ω as $\Omega = \sum_{n=1}^{\infty} \Omega_n$. Please note that probabilities are $\sigma-$finite measures.

Definition 4. *Borel sets and Lebesgue Measure*
Let $I = \{(a, b], (-\infty, b], (a, \infty) : a, b \in \mathbb{R}\}$. Please note I is not a field. Let $\mathcal{B}_I$ be the collections of sets consisting of all finite disjoint unions of elements of I. So $\mathcal{B}_I$ is a field. Define $\mathcal{B} \equiv \sigma[\mathcal{B}_I]$. $\mathcal{B}$ is called the Borel sets of $\mathbb{R}$. For each $A \in \mathcal{B}_I$, define $\lambda(A) = (\sum_{i=1}^{n} \lambda(A_i))$, where $A = (\sum_{i=1}^{n}(A_i))$ with $A_i \cap A_j = \emptyset \forall i, j \in 1, 2, ..., n$ and $\lambda(A) \equiv$ length of the set A.

Proposition 2. *Monotone property of measures*
Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Let $(A_1, A_2, ...)$ be in $\mathcal{A}$.

- *If $A_n \subset A_{n+1} \forall n$ i.e. increasing sequence of sets, then $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.*
- *if $\mu(A_{n_0}) < \infty$ for some n_0, and $A_{n+1} \subset A_n \forall n$ i.e. decreasing sequence of sets, then $\mu\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.*
- *Whenever $(A_1, A_2, ...)$ and $\cup_{n=1}^{\infty} A_n$ are all in $\mathcal{A}$, then $\mu\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$. This also holds for a measure on a field.*

Theorem 1. *Carathéodory Extension Theorem*
A measure μ on a field $\mathcal{C}$ can be extended to a measure on the $\sigma-$field $\sigma-[\mathcal{C}]$ generated by $\mathcal{C}$.

A probability measure ($\sigma-$finite measure) defined on a field $\mathcal{C}$ has an unique extension[2] to the $\sigma-$field $\sigma - [\mathcal{C}]$ generated by $\mathcal{C}$. From this we can extend the Lebesgue measure from $\mathcal{B}_I$ to $\mathcal{B}$. This extension is also called the Lebesgue measure.

Definition 5. *A measure μ on the real line $\mathbb{R}$ assigning finite values to finite intervals is called a Lebesgue-Stieltjes measure.*

Definition 6. *Generalized density function*
A finite increasing function F on $\mathbb{R}$ that is right-continuous is called a generalized density function (gdf). Then $F_{-}(\cdot) \equiv \lim_{y \nearrow \cdot} F(y)$ denotes the left-continuous version of F. The mass function of F is defined by $\Delta F(\cdot) = F(\cdot) - F_{-}(\cdot)$, while $F(a, b] = F(b) - F(a) \ \forall \ a < b$ is called the increment function of F.

Theorem 2. *The correspondence theorem*
Let F be defined as in definition 6. The relationship $\mu((a, b]) = F(a, b]$ for all $-\infty \leq a < b \leq +\infty$ establishes a $1 - to - 1$ correspondence between the Lebesgue-Stieltjes measures μ on $\mathcal{B}$ (i.e. on $(\mathbb{R}, \mathcal{B})$) and the representative members of the equivalence classes of gdfs.

[2] The unique extension exists when the measure is $\sigma-$finite.

For Lebesgue measure λ, a gdf is the identity function $F(x) = x$. In the general model defined below, we use the correspondence theorem without any reference.

Proposition 3. *A function* $g : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ *is measurable if and only if* $g^{-1}((-\infty, b]) \in \mathcal{A}$ *for all* $b \in \mathbb{R}$.

Proposition 4. *A function* $Q : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ *is a transition function if for all* $\omega \in \Omega$, $Q(\omega, \cdot)$ *is a probability measure and if given* $A \in \mathcal{A}$, $Q(\cdot, A)$ *is a measurable function.*

Proposition 5. *A measure* λ^* *is an invariant distribution with respect to the transition function* Q *if for all* $A \in \mathcal{A}$, $\lambda^*(A) = \int_A Q(\omega, A) d\lambda^*$.

We also use the Lebesgue measure on the product space. This measurable product space is constructed in an analogous way as in $\mathbb{R}^1$. Please see reference [11] for this construction. For now, we have given enough background to discuss the following model.

1.2 General Model

The model that we consider is a version of the economy described in [7] and [15]. Consider an economy with a large number of infinitely-lived consumers subject to individual endowment shocks and a single firm subject to aggregate productivity shocks in every period. By shocks we mean uninsurable uncertainty. In this paper individual or idiosyncratic endowment shocks will mean uncertainty in employment, and aggregate shocks uncertainty at the economic level (crop failure or boost, technological failure or advancement, etc). Time is discrete and denoted by $t = 0, 1, 2, ...$. Uncertainty is represented by a probability product space $(\Omega \times \mathbb{Z}^\infty, \mathcal{F}, \mathbf{P})$ on which stochastic processes are defined. The state space Ω captures individual shocks, while the state space $\mathbb{Z}^\infty$ captures aggregate shocks. Let $\mathbb{Z}^0 = \mathbb{Z}$, $\mathbb{Z}^{t+1} = \mathbb{Z} \times \mathbb{Z}^t$, and $z^t = (z_0, z_1, ..., z_t) \in \mathbb{Z}^t$ an aggregate shock history at time t. Likewise $z^\infty = (z_0, z_1, ...) \in \mathbb{Z}^\infty$ is the complete history and $z^0 = z_0 \in \mathbb{Z}^0$. Assume $\mathbb{Z} \subset [\underline{z}, \bar{z}] \subset \mathbb{R}_{++}$ (strictly positive space) is a bounded and countable set endowed with a discrete topology. $\mathbf{P}$ represents a probability measure and $\mathcal{F}$ a σ-field defined on the product space.

1.2.1 Consumers

Consumers or households or agents or individuals (all used interchangeably) are distributed on a continuous interval. In this paper, we use one continuous interval, but in general, Borel subsets of $\mathbb{R}$ can be used. This is to allow the use of Lebesgue measure to count the households. Consumers are ex ante identical in that they have the same preferences and their endowment shock processes are drawn from the same distribution. However, consumers are ex post heterogeneous in the sense that they experience idiosyncratic employment shocks. Households save in good (employed) times and run down their wealth in bad (unemployed) times, and hence, also vary in asset wealth.

Information structure and endowments: Consumer $i \in I$ is endowed with one unit of labor at the beginning of each period t and a deterministic asset level $a_0^i \in \mathbb{R}_{++}$ at the beginning of period 0. Labor endowment is subject to random shocks represented by a stochastic process $(\epsilon_t^i)_{t \geq 0}$ valued in $\mathbb{S} \subset \mathbb{R}_+$, where ϵ_0^i is a deterministic constant. Let $\mathbb{S}^0 = \mathbb{S}$, $\mathbb{S}^{t+1} = \mathbb{S}^0 \times \mathbb{S}^{t+1}$, and $\epsilon^{ti} = (\epsilon_0^i, \epsilon_1^i, ..., \epsilon_t^i)$ with $\epsilon^{0i} = \epsilon_0^i$. Assume $\mathbb{S} \subset \mathbb{R}_+$ is compact. Let the initial probability distribution of asset holdings and endowment shocks be given by

$$\Gamma_0(A \times S) = \mu(i \in I : (a_0^i, \epsilon_0^i) \in A \times S), (A \times S) \in (\mathcal{B}(\mathbb{R}_{++}) \times \mathcal{B}(\mathbb{S})). \tag{1.1}$$

At the beginning of time t, consumer i observes his labor endowment shock ϵ_t^i and the aggregate productivity shock, z_t. His information is represented by a σ-field $\mathcal{F}_t^i$ generated by the information of the past and current shocks $\{\epsilon_n^i, z_n\}_{n=0}^t$. Assumptions on the shock processes are given below to maintain measurability.

Assumption

- Given the history $(\epsilon^{it}, z^t) = (\epsilon^t, z^t)$, $(\epsilon_{t+1}^i, z_{t+1})$ is drawn from the distribution $Q_{t+1}(\cdot, \epsilon^t, z^t)$ for all $i \in I$;
- $Q_{t+1}(\mathcal{S} \times \mathcal{Z}, \cdot)$ is measurable for all $\mathcal{S} \times \mathcal{Z} \in \mathcal{B}(\mathbb{S}) \times \mathcal{B}(\mathbb{R})$
- Q_{t+1} has the Feller property: $\int h(\epsilon', z') Q_{t+1}(d\epsilon', dz', \cdot)$ is a continuous function on $\mathbb{S}^t \times \mathbb{Z}^t$ for any real-valued, bounded and continuous function h on $\mathbb{S} \times \mathbb{Z}$.

Consumption space: There is a single good. A consumption plan $c^i \equiv (c_t^i)_{t=0}^\infty$ for consumer i is a nonnegative real-valued process such that c_t^i is $\mathcal{F}_t^i$-measurable. Let $\mathcal{C}^i$ denote the set of all consumption plans for consumer i.

Budget and borrowing constraints: An asset accumulation plan $(a_{t+1}^i)_{t\geq 0}$ for consumer i is a real-valued process such that a_{t+1}^i is $\mathcal{F}_t^i$-measurable.

In each period t, consumer i consumes c_t^i and accumulates assets a_{t+1}^i subject to the budget constraint:

$$c_t^i + a_{t+1}^i = (1 + r_t) a_t^i + w_t s_t^i \ , \ \text{ given } a_0^i. \tag{1.2}$$

$$a_{t+1}^i \geq 0, \ \ \forall i \in I. \tag{1.3}$$

Let $\mathbb{A} = [0, \infty)$, and $\mathcal{A}^i$ denote the set of all asset accumulation plans for consumer i that satisfy 1.2 and 1.3. Equation 1.3 implies no borrowing. This simplifies the model so we can ignore bonds and loan defaults. A consumption plan $c \in C^i$ corresponding to an asset accumulation plan $a \in \mathcal{A}^i$ is called (budget) feasible.

Preferences: Consumer i's preferences are represented by an expected utility function defined on C^i:

$$U(c^i) = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t^i)\right], \ (c_t^i) \in \mathcal{C}^i, \tag{1.4}$$

where $\beta \in (0, 1)$ is the discount factor, and the following assumption holds.

Assumption on consumer utility: $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, is a bounded continuously differentiable, strictly increasing and strictly concave function with $\lim_{c \to 0} u'(c_t) = \infty$.

In this paper, we will use the following constant relative risk aversion (CRRA)[3] utility function:

$$u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta} & \text{if } \ \eta > 0 \ \& \ \eta \neq 1 \\ \ln(c) & \text{if } \eta = 1 \end{cases} \tag{1.5}$$

[3]This function is commonly used in literature. η is called the coefficient of relative risk aversion and is typically set to 2.

Decision problem: Consumer i's problem is given by:

$$\sup_{(c_t^i, a_{t+1}^i)_{t\geq 0} \in C^i \times \mathcal{A}^i} U(c^i). \tag{1.6}$$

The plans $(c_t^i)_{t\geq 0}$ and $(a_{t+1}^i)_{t\geq 0}$ are optimal if the above supremum is achieved by $(c_t^i, a_{t+1}^i)_{t\geq 0} \in C^i \times \mathcal{A}^i$.

Allocation: An allocation $((c_t^i, a_{t+1}^i)_{t\geq 0})_{i\in I}$ is a collection of consumption and asset accumulation plans. An allocation is admissible if both $c_t^i = c_t(i, \omega, z^t)$ and $a_{t+1}^i = a_{t+1}(i, \omega, z^t)$ are $\mathcal{B}(I) \otimes \mathcal{F}_t$ - measurable where $\mathcal{F}_t$ is the smallest σ-algebra containing $\mathcal{F}_t^i$ $\forall i \in I$, $\mathcal{F}_t = \bigvee_{i\in I} \mathcal{F}_t^i$, $t \geq 0$. Since c_t^i and a_{t+1}^i are $\mathcal{F}_t^i$-measurable $\forall i \in I$, they are also $\mathcal{F}_t$-measurable.

1.2.2 The Firm

There is a single firm renting capital at rate r_t and hiring labor at wage w_t at time t from the households. It produces output

$$Y_t = z_t F(K_t, N_t) + (1-\delta)K_t, \tag{1.7}$$

where $\delta \in [0, 1]$ is the capital depreciation rate. Aggregate capital K_t is $\mathcal{F}_{t-1}$ measurable, and aggregate labor N_t is $\mathcal{F}_t$ measurable.

Assumption on Labor and Production: N_t is uniformly bounded, $0 \leq N_t \leq \hat{N}$. The production function F is homogeneous of degree one, strictly increasing, strictly concave, continuously differentiable, and satisfies: $F(0, \cdot) = 0, F(\cdot, 0) = 0$, $\lim_{K\to 0} F_K(K, \hat{N}) = \infty$, and $\lim_{K\to\infty} F_K(K, \hat{N}) = 0$.

Throughout this paper, we will set $F(K_t, N_t)$ to be

$$F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}\ ,\ \alpha \in (0, 1). \tag{1.8}$$

Firm's Decision problem: The firm has to choose optimal K_t and N_t to maximize profits based on the following problem:

$$\sup_{(K_{t+1}, N_t)} Y_t - r_t K_t - w_t N_t. \tag{1.9}$$

given w_t and r_t. Since the households collectively determine how much capital K_t and labor N_t to supply using 1.6, the firm's problem becomes a static one. The firm's optimization problem implies the following marginals:

$$r_t = z_t \frac{\partial F}{\partial K_t}(K_t, N_t) - \delta \tag{1.10}$$

and

$$w_t = z_t \frac{\partial F}{\partial N_t}(K_t, N_t), \tag{1.11}$$

where the factor prices r_t and w_t are $\mathcal{F}_t$ - measurable.

1.2.3 Competitive Equilibrium

A sequential competitive equilibrium consists of an admissible allocation $((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}$ and price processes $(r_t, w_t)_{t\geq 0}$ such that:

1. Given prices $(r_t, w_t)_{t\geq 0}$, $(a_{t+1}^i, c_t^i)_{t\geq 0}$ solves problem 1.6 for μ-a.e. i.
2. Given prices $(r_t, w_t)_{t\geq 0}$, the firm maximizes profits so that 1.10 and 1.11 are satisfied for all $t \geq 0$.
3. For all $t \geq 0$, the labor market clears

$$\int_I \epsilon_t^i \mu(di) = N_t,$$

and the goods market clears

$$C_t + K_{t+1} = z_t F(K_t, N_t) + (1-\delta)K_t,$$

where $C_t = \int_I c_t^i \mu(di)$ and $K_t = \int_I a_t^i \mu(di)$.

An aggregate distribution is defined over the individual states across the population. An individual state is a pair of individual asset holdings and the history of individual shocks. If individual asset holdings and the shock history at date $t \geq 0$ are a_t^i and ϵ^{ti} for $i \in I$, respectively, then the aggregate distribution, $\Gamma_t \in \mathcal{P}(\mathbb{A} \times \mathbb{S}^t)$, is defined by

$$\Gamma_t(A \times B) = \mu(i \in I : (a_t(i), \epsilon^t(i)) \in A \times B), \ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}^t). \tag{1.12}$$

Thus Γ_t is a random measure as $a_t^i = a_t^i(\omega, z^{t-1})$ and $\epsilon_t^i = \epsilon_t^i(\omega, z^t)$ are random variables with $(\omega, z^t) \in \ \Omega \times \mathbb{Z}^t$.

Aggregate variables are written as expectations with respect to the so defined aggregate distribution:

$$\begin{aligned}
K_t = & \ \int_I a_t^i \mu(di) = \int_{\mathbb{A}\times\mathbb{S}^t} a_t \Gamma_t(da, d\epsilon^t), \\
N_t = & \ \int_I \epsilon_t^i \mu(di) = \int_{\mathbb{A}\times\mathbb{S}^t} \epsilon_t \Gamma_t(da, d\epsilon^t), \\
C_t = & \ \int_I c_t^i \mu(di) = (1+r_t)K_t + w_t N_t - K_{t+1},
\end{aligned}$$

where the last equation is obtained by substituting the budget constraint in 1.2 for c_t^i. Equations 1.10 and 1.11 induce the pricing functions $r_t : \mathcal{P}(\mathbb{A} \times \mathbb{S}^t) \times \mathbb{Z} \to \mathbb{R}$ and $w_t : \mathcal{P}(\mathbb{A} \times \mathbb{S}^t) \times \mathbb{Z} \to \mathbb{R}_+$ as follows:

$$r_t(\Gamma_t, z_t) = z_t F_1 \left(\int_{\mathbb{A}\times\mathbb{S}^t} a_t \Gamma_t(da, d\epsilon^t), \int_{\mathbb{A}\times\mathbb{S}^t} \epsilon_t \Gamma_t(da, d\epsilon^t) \right) - \delta,$$

$$w_t(\Gamma_t, z_t) = z_t F_2 \left(\int_{\mathbb{A}\times\mathbb{S}^t} a_t \Gamma_t(da, d\epsilon^t), \int_{\mathbb{A}\times\mathbb{S}^t} \epsilon_t \Gamma_t(da, d\epsilon^t) \right).$$

Competitive equilibrium is difficult to achieve even for a simple economy like the one presented above. Furthermore, redefining the expected utility 1.6 as a dynamic programming problem for more complex models is nontrivial. The solution of interest is not just a competitive equilibrium but a stationary equilibrium. In a stationary equilibrium, the aggregate variables have converged. For instance, for the aggregate capital stock this implies $\lim_{t \to \infty} K_t \to K^*$. Stationary equilibrium solutions take a long time to converge if they exist at all. In all the models discussed in this paper, two factors are of prime interest (i) how to achieve stationary equilibrium (with a main focus on convergence of aggregate capital) and (ii) the running time to obtain this potential solution.

The structure of this paper is as follows. Chapter 2 gives background on existing methods to solve heterogeneous economies. Chapter 3 introduces a homogeneous agent based economy. This chapter conceptually builds on the nature of the expected solution. Chapter 4 discusses a heterogeneous agent economy with idiosyncratic shock. Algorithms are discussed in detail, and our approach to find the stationary density is introduced. We also introduce a method to find the solution using stochastic simulation. In this case, the history becomes part of the consumer's state space. Since this is a common approach, we have described the general model above using the stochastic simulation - time iterative - approach. Chapter 5 discusses the final model - an economy with both aggregate and idiosyncratic shocks. We end with a discussion on the future direction of this work in chapter 6.

CHAPTER 2

CURRENT LITERATURE

Models with heterogeneous agents and incomplete markets[1] are becoming exceedingly important in macroeconomics and finance. As models get richer, solutions become harder. This is because the household state space includes a cross-sectional distribution (referred to as Γ in equation 1.12) which is a high dimensional object. In addition, the problems are highly nonlinear.

The model defined in chapter 1 is relatively simple but nontrivial. There are of course more complex models considered in the literature. The fact that different algorithms generate non-similar results motivates us to be careful in solving these models numerically.[2] In this paper, we present an algorithm that is robust across models. In the upcoming chapter, we develop the nature of the expected solution and present our method to solve a model of a heterogeneous economy with idiosyncratic risk. We present a comparative analysis of the accuracy and speed of existing algorithms summarized in [4] for this type of economy. We then use our algorithm to solve another heterogeneous economy with aggregate and idiosyncratic risk. Our technique is a fixed point iteration algorithm. In the literature, time iteration techniques[3] are generally believed to be faster and more reliable than fixed point iteration. In chapter 4, we contrast this by presenting an accurate and a reasonably fast solution. Furthermore, we explain the failures reported by [4] in solving a model with aggregate certainty and idiosyncratic risk.

Recursive numerical solutions of heterogeneous economies consist of functions of state variables. Existing algorithms are based on either projection or perturbation methods or both. The projection method consists of two steps. In the first step, a grid of state variables is constructed and one defines at each grid point error terms that provide a measure for the fit of any approximating function. Numerical procedures, such as quadrature methods to calculate conditional expectations, may still be needed to calculate the value of the error terms. The second step consists of choosing the coefficients of the numerical approximation to obtain the best fit for the given loss function of the error terms.

To solve the general model presented in chapters 1 and 5, different algorithms follow different techniques in summarizing the cross-sectional distribution of capital and employ-

[1] In economics, a complete market is one in which the complete set of possible gambles on future states-of-the-world can be constructed with existing assets. In incomplete markets this is not possible as the gambles have to be determined sequentially as the probabilistic state in the economy unfolds.

[2] Only a handful of models have an analytical solution.

[3] The general model presented in chapter 1 is set up using this technique.

ment status with limited moments. These were first introduced by [7], [3] and [8]. The Krusell and Smith [7] algorithm specifies a law of motion for these moments and finds the approximating function using a simulation procedure. That is, given a set of policy rules, a time series of cross-sectional moments is generated and new laws of motion for aggregate moments are estimated using the simulated data. In particular, they only consider the simple case that agents only use the first moment, i.e. the aggregate capital stock, of the cross-sectional distribution. They find that the forecast error due to the omission of the higher moments is extremely small. An important advantage of the stochastic-simulation Krusell and Smith algorithm is that it is simple, intuitive and easy to program. As Algan et al. [20] show, however, stochastic-simulation methods have two potential shortcomings. First, the introduction of stochastic simulations produces sampling noise, which makes the policy rules depend on a specific random draw. Second, the simulated endogenous data are clustered around the mean, which implies that the accuracy of the approximation on the tails is low. They argue that replacing a stochastic simulation with a non-stochastic one can enhance the accuracy and speed of the algorithm. Therefore, it is of interest to assess the accuracy of the stochastic-simulation version of the Krusell and Smith algorithm and compare it with a non-stochastic-simulation version. Maliar et al. [18] does this and reports that the Krusell and Smith method produces sufficiently accurate solutions. They further simulate the economy using a finite number of agents while Young [19] uses a numerical procedure to simulate a continuum of agents. Given the aggregate law of motion, the laws of motion of the individual variables are then updated using standard projection methods.

Existing algorithms are based on either the projection method or the perturbation method, sometimes on both. The projection method consists of two steps. In the first step, a grid of the state variables is constructed and one defines at each grid point error terms that provide a measure for the fit of any approximating function. The second step consists of choosing the coefficients of the numerical approximation to obtain the best fit for a given loss function of the error terms. The perturbation approach solves for the coefficients of the Taylor expansion of the true set of policy functions $h(x)$ around the steady state. Using $h(x)$ the choice variables can be substituted out of the model equations and one obtains a system of equations with x as the only variable, that is, $F(x) \equiv 0$. The unknown coefficients of the Taylor expansion are found by sequentially differentiating this system of equations and evaluating the obtained equations at the steady state. A brief overview of current literature on projection and perturbation methods is given below.

Projection Method 1 - Parameterization of the cross-sectional distribution: Reiter [12] and Algan et al. [20] solve the model using projection. [12] uses a technique that involves both projection and perturbation (see below) and computes a solution that is fully nonlinear in the idiosyncratic shocks, but linear in the aggregate shocks. [20] uses parameterization of the cross sectional distribution, but allows for more flexibility without increasing the number of state variables. They both obtain the next period's cross sectional moments by explicitly integrating the individual choices instead of using simulation. They assume a functional form for the cross sectional distributions and solve for the coefficients of the approximating distribution via simulation. The time varying coefficients become state variables. This allows them to avoid a disadvantage of the Krusell and Smith [7] algorithm, namely the points at which the aggregate law of motion is determined are chosen inefficiently. The Krusell and Smith algorithm relies on the idea that the next period's moments are perfectly forecastable, which is at best approximately true in a simulation with a finite number of agents. One problem with the approaches in [12] and [20] is that it is not always clear how to construct a sensible density. And describing the cross sectional density requires several coefficients, and this increases the individual state space significantly.

Projection Method 2 - No Parameterization of the cross-sectional distribution: Den Haan et al. [14] derive the aggregate laws of motion directly from the individual policy rules by simply aggregating them. They avoid numerical integration by writing the individual policy functions as linear combinations of basis functions of the individual state variables. This analytic approach requires including the cross sectional averages of all the basis functions that enter the individual policy functions. This infinitely dimensional object is also an individual state variable. Algan et al. [20] propose an algorithm that uses projection methods and can - in principle - solve the model without relying on any simulation procedure. Using projection procedures to solve a model with a continuum of agents typically requires a parameterization of the cross-sectional distribution as in [3]. Algan et al. improve upon the projection procedure by allowing more general approximating functions with more free parameters without increasing the individual state space. For instance, if one uses a Normal density then there are two parameters, i.e., the mean and the variance, and thus two state variables. But note that using a Normal density has implications for the higher-order moments. These implied higher-order moments may not be correct. For example, a Normal density implies no skewness, but the model to be solved may have a skewed distribution. In that case one could allow for more general approximating functions with more free parameters. The information obtained from the simulation is used to modify the functional form of the cross-sectional distribution. The underlying philosophy is the same as in [12], but the implementation is less cumbersome.

Perturbation Methods: The approximating functions used in perturbation techniques are continuous and differentiable and consequently, are not suited for the models presented in this paper. Preston and Roca [21] solve a model similar to the general model considered in this paper. They replace the borrowing constraints with a penalty function and the finite-state Markov process with a stochastic process with continuous support. Kim et al. [22] use the same approach but instead of perturbation they use the solution from a deterministic economy (without uncertainty) without any constraint or penalty functions. This approach optimizes computation time but makes the solution irrelevant to the target model.

Computational Speed: There are enormous differences in speed between the algorithms discussed above. Perturbation algorithms are likely to outperform projection methods in speed. So one needs to be careful in making a comparison across various techniques. The existing time iteration methods have a wide range of computational speeds. The algorithms introduced in this paper use a fixed point iteration method. Although a valid comparison between this method and a time iteration approach may not be valid, we briefly comment on the running time to put things into perspective. Our goal in this study is to present a robust, reproducible and relatively accurate solution with reasonable computing time.

CHAPTER 3

A SINGLE CONSUMER AND PRODUCER DETERMINISTIC ECONOMY

Consider a simple model with a single consumer and producer economy. Assume for now that there is no uncertainty - everything is deterministic, and this single person economy is closed. Think of a farmer producing a commodity (say corn) for consumption in the current period and for reproduction (saving seed to replant) in the next period. Let production be defined by a deterministic function, $F(k_t, n_t)$, where k_t and n_t are defined as amount of input seed and labor, respectively.
Assumption on Labor and Production: n_t is uniformly bounded, $0 \le n_t \le \hat{n}$. The production function, $F(k_t, n_t)$, is assumed to have a simple analytic form. It is homogeneous of degree one, strictly increasing, strictly concave, continuously differentiable, and satisfies: $F(0, \cdot) = 0, F(\cdot, 0) = 0, \lim\limits_{k \to 0} \frac{\partial F}{\partial k}(k, \hat{n}) = \infty$, and $\lim\limits_{k \to \infty} \frac{\partial F}{\partial k}(k, \hat{n}) = 0$.
Throughout this paper, we will use the Cobbs-Douglas production function:

$$F(k_t, n_t) = D k_t^{\alpha} n_t^{1-\alpha} \ , \ \alpha \in (0, 1), \tag{3.1}$$

where D is a constant.
The farmer's asset (corn/seed) at the end of the period t can be given as:

$$y_t = F(k_t, n_t) + (1 - \delta) k_t, \tag{3.2}$$

where $(1 - \delta) k_t$ is the seed saved in period $t - 1$ for reproduction. The value of seed is depreciated using a constant $\delta \in (0, 1)$. This is not only because the value of seed decreases over time, but also to build a conceptual analogy to wealth in the models to come. Let's assume labor supply is unconstrained and the farmer supplies 100% of it.
Thus, we set the labor supply to be a constant, 1 [1], and obtain:

$$y_t(k_t) = y_t := F(k_t, 1) + (1 - \delta) k_t$$

[1] $0 \le n_t \le 1$. Leisure is not valued and $F_2(k_t, n_t)$ is an increasing function. So the optimal occurs at $n_t = 1 \; \forall \, t$.

The following constraints in period t are then implied:

$$\begin{aligned} c_t + k_{t+1} &\leq y_t && (3.3)\\ c_t + k_{t+1} &\leq F(k_t, 1) + (1-\delta)k_t && (3.4)\\ 0 &\leq k_{t+1} \leq y_t && (3.5)\\ 0 &\leq c_t && (3.6)\\ 0 &\leq k_t, && (3.7) \end{aligned}$$

where c_t, k_{t+1} are the optimal consumption and future savings choices, respectively, that the farmer has to make. The farmer's choice of consumption ${c_t}^2$ is aimed at maximizing the lifetime[3] utility below:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (3.8)$$

$$\Rightarrow \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(y_t - k_{t+1}), \quad (3.9)$$

such that the constraints in 3.3 are satisfied.

The utility function, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, is bounded continuously differentiable, strictly increasing and strictly concave function with $\lim_{c \to 0} u'(c_t) = \infty$. In this paper we use the constant relative risk aversion (CRRA) utility:

$$u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta} & \text{if } 0 < \eta < 1 \ \& \ \eta > 1 \\ \ln(c) & \text{if } \eta = 1 \end{cases}, \quad (3.10)$$

where η is called the coefficient of relative risk aversion.

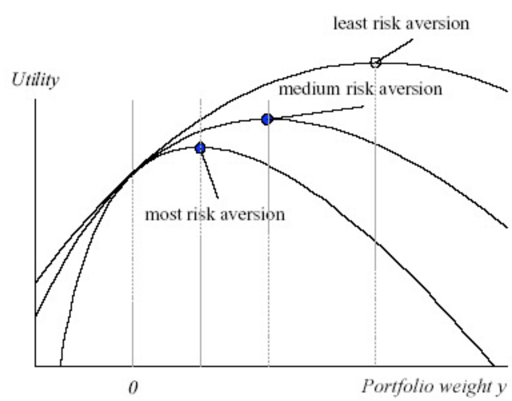


Figure 3.1: Utility plot of a portfolio that contains one consumption good.

[2]We assume the interior solution so that $c_t > 0$ implies $c_t = -k_{t+1} + (1-\delta)k_t + F(k_t, 1)$.

[3]The farmer optimizes thinking he/she will live for infinite time and hence the sum is to infinity. This is called an infinite time horizon problem.

The utility can be seen as a preference curve and describes the willingness of the consumer to forgo consumption today for a later time. That is, if $c_t < c_{t+s}$ with $t, s > 0$ and $u(c_t) > u(c_{t+s})$, then the consumer prefers to consume at time t over $t + s$. In general, a higher the value of η implies greater the risk aversion. Please see Figure 3.1 [4] for a brief description. The portfolio, y (not related to production defined earlier), in the figure has one consumption good. *Setting the lifetime utility recursively*: The farmer's lifetime utility can be expressed recursively[5] as follows:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.11}$$

$$\Rightarrow \quad \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(y_t - k_{t+1}) \tag{3.12}$$

$$\Rightarrow \quad V^*(k_t) = \max_{0 \le k_{t+1} \le y_t} \{u[y_t - k_{t+1}] + \beta V^*(k_{t+1})\}. \tag{3.13}$$

The above process is called dynamic programming, a method of solving complex problems by breaking them down into simpler steps. This enables us to view maximization over infinite periods as a one step optimization problem. Restating the problem without time subscripts we obtain: the function $V^* : \mathbb{R}_+ \to \mathbb{R}$ as the limit of the following sequence of steps

$$V^{s+1}(k) = \max_{0 \le k' \le y} \{u[y - k'] + \beta V^s(k')\}, \tag{3.14}$$

where the next period values are expressed using the 'prime' notation. Of course, V^* is an unknown function so far, but it is differentiable, strictly increasing and strictly concave. [10] develops the existence and uniqueness of such a V for all the models discussed in this paper.

The existence and uniqueness of such a V^* is derived from the Contraction Mapping Theorem:

Theorem 3. *If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then*

- *T has exactly one fixed point V in S, and*
- *for any $V_0 \in S$, $\rho(T^n V_0, V) \le \beta^n \rho(V_0, V)$, $n = 1, 2, 3, ..$*

See [10] for a proof.

In our case, successive iterations in the dynamic programming leads us to the limiting point in the function space, V^*, of interest. Once we find the converged value function, V^*, we are interested in determining the optimal policy. The sequence of $\{k_{t+1}\}_{t=0}^{\infty}$ that we obtain is called the optimal policy, and the solution[6] $k_{t+1} = h(k_t)\ \forall t$ is the optimal policy function. We demonstrate this using an example.

[4] Taken from http://mitocw.udsm.ac.tz/OcwWeb/Sloan-School-of-Management/15-433InvestmentsSpring2003/CourseHome/index.htm

[5] Functions considered in this paper can all be stated recursively.

[6] $h : \mathbb{R}_+ \to \mathbb{R}$ is increasing and differentiable.

Example 1 (The nature of solution in the deterministic model)**.** *Let's consider an example where the one-period utility function u with $\eta = 1$ and the production function F is given with a constant labor input of 1. Please also note that we ignore the time subscripts.*

$$u(c) = \ln c$$
$$F(k) = k^{\alpha},\ \alpha \in (0,1)$$
$$y(k) = F(k) + (1-\delta)k,$$

respectively. Let α and β be 0.3500 and 0.9844, respectively. If we set $\delta = 1$ and $\eta = 1$ (log utility), [4] shows there exists a closed form solution to 3.14. The proof is reproduced in detail below:

Analytical Equilibrium solution to the deterministic model:
Let $v^0 = 0$, $u(c) = \ln(c)$, $F(k) = k^{\alpha}$, $\alpha \in (0,1)$ and $f(k) = F(k) + (1-\delta)k$ with $\delta = 1$. Recall $c = f(k) - k'$, and we solve for the one period value function as follows:

$$\begin{aligned} v^1(k) &= \max_{0\le k'\le f(k)} \{u[k^{\alpha} - k'] + \beta v^0(k')\} \\ v^1(k) &= \alpha \ln(k). \end{aligned}$$

This implies $k' = 0$ and $v^1(k) = \alpha \ln(k)$. For the next step, we get:

$$\begin{aligned} v^2(k) &= \max_{0\le k'\le f(k)} \{u[k^{\alpha} - k'] + \beta v^1(k')\} \\ &= \max_{0\le k'\le f(k)} \{u[k^{\alpha} - k'] + \beta\alpha \ln(k')\} \end{aligned}$$

taking derivative with respect to k' in $v^2(k)$ and setting equal to zero, we obtain:

$$\frac{1}{k^{\alpha} - k'} = \frac{\alpha\beta}{k'}.$$

Solving for k' gives

$$\begin{aligned} k' &= \frac{\alpha\beta}{1+\alpha\beta} k^{\alpha} \\ v^2(k) &= \alpha(1+\alpha\beta)\ln(k) + D_1 \\ D_1 &= \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right). \end{aligned}$$

For $v^3(k)$, we get

$$\begin{aligned} v^3(k) &= \max_{0\le k'\le f(k)} \{u[k^{\alpha} - k'] + \beta v^2(k')\} \\ &= \max_{0\le k'\le f(k)} \{u[k^{\alpha} - k'] + \beta(\alpha(1+\alpha\beta)\ln(k') + D_1)\}. \end{aligned}$$

Taking derivative with respect to k', and solving for k', we get:

$$k' = \frac{\alpha\beta + (\alpha\beta)^2}{1 + \alpha\beta + (\alpha\beta)^2} k^{\alpha}.$$

Continuing in similar manner, we get

$$k' = \frac{\sum_{i=1}^{s-1} (\alpha\beta)^s}{\sum_{i=0}^{s-1} (\alpha\beta)^s} k^\alpha$$

and taking limit $s \to \infty$, we get

$$k' = \alpha\beta k^\alpha. \tag{3.15}$$

that is

$$\begin{aligned} k_{t+1} &= h(k_t) \\ &= \alpha\beta K_t^\alpha. \end{aligned}$$

The point $h(k) = k$ is called the stationary solution, steady state or the fixed point of h. In this example, there is one positive stationary point, $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.194$, i.e. $\lim_{t\to\infty} k_t \to k^*$.

The numerical solution[7] is computed fairly easily using the following algorithm.

Algorithm 1 (Numerical solution to the deterministic model in example 1)**.**

1. *Initialize the value function to $V_0 = 0\ \forall i = 1...n$ where n is the number of grid points*[8].
2. *Compute $V^{s+1} = \max_{k_j}\{u(y(k_i) - y(k_j)) + \beta V^s\}$ for $j = 1, ..., n$ for the s^{th} iteration $\forall$ grid points. So V^s is a vector.*
3. *Store the grid location, j^*, of k' in a policy vector, h.*
4. *Repeat steps 3 and 4 until the norm of two consecutive V's is within some $\epsilon > 0$, our termination condition.*

Figure 3.2 plots k_t versus k_{t+1} using the policy generated by dynamic programming along with the closed form solution. The point of intersection between the 45° line and the policy curve is the equilibrium point, $k^* \approx 0.194$, as defined in example 1. The running time for the above implementation in MATLAB on a 2.4 GHz processor is about 19 seconds. We could further improve on the running time by exploiting the monotonicity of the value function, i.e. in step 3, once we find the optimal index j_1^* for k_1 we only need to consider capital stocks greater than $k_{j_1^*}$ in the search for the next optimal index j_2^* for k_2. It is important to note that the maximum of the value function need not occur on the grid points, and to account for this we use linear interpolation throughout this paper.[9] In this deterministic case the solution on and off the grid computed using interpolation is about

[7] An alternative approach could be that of the Lagrangian: taking derivatives and setting the first order conditions. Since we are setting a premise to solve higher dimension problem that have only numerical solutions, we develop the dynamic programming approach and solve all examples using it in this paper.

[8] A finer grid leads to better approximation, but comes at a cost - longer running time.

[9] REITER2009 uses cubic spline interpolation.

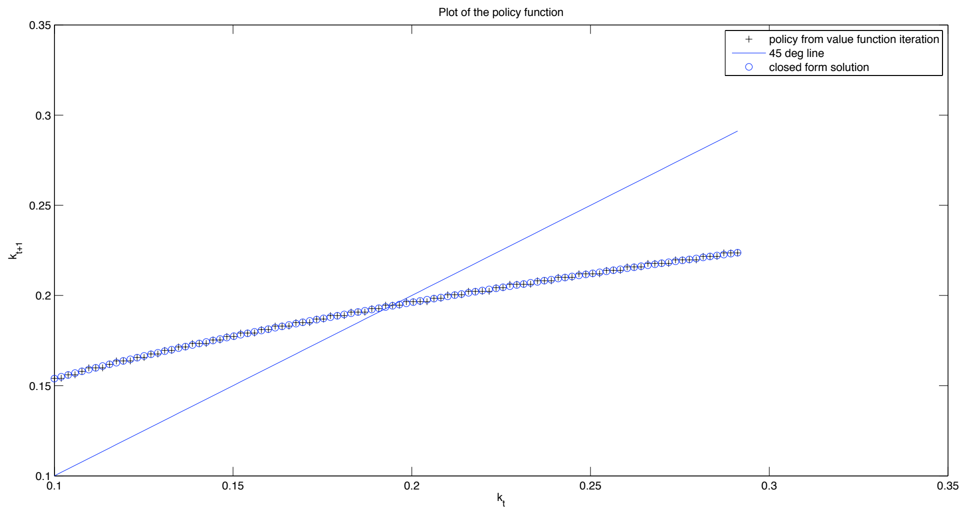


Figure 3.2: Capital at time t versus capital at time $t+1$ using the policy function generated by dynamic programming and the closed form solution.

the same (plot not shown here); however, the error is amplified in subsequent calculations in later models with a higher order state space.[10]

For our second model we introduce aggregate shock (crop failures, technological breakthroughs, etc at the national level) to the current model and show the equilibrium shifts from a point to a limiting distribution.

3.1 A Single Consumer and Producer Aggregate Shock Economy

This model is also called a stochastic model for optimal growth. The model is the same as in the last section, but with a minor difference - the farmer's production of corn is affected by aggregate uncertainty (crop failure, sudden mutation to seed that causes production boost, etc). The production function, $F(k_t, n_t)$, will have a stochastic term, z_t, where $\{z_t\}$ is a sequence of independent and identically distributed random variables. $z_t > 1$, $z_t = 1$, $0 \leq z_t \leq 1$ imply economic boost, no shock, and economic downturn, respectively.
Again we assume labor is supplied maximally and set it to a constant 1 (i.e. $n_t = 1$).
At the beginning of period t the current value z_t of the exogenous shock is realized, thus $z_t F(k_t, 1)$ is known. The farmer has to decide how much to consume in the current period c_t and how much to save for the next period k_{t+1}.

[10] This is when two or more arguments are passed into the policy function.

In period t, the farmer solves the following maximization problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{3.16}$$

$$\Rightarrow \max_{\{k_{t+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u((1-\delta)k_t + z_t F(k_t, 1) - k_{t+1})\right] \tag{3.17}$$

using the constraint equation:

$$c_t + k_{t+1} = (1-\delta)k_t + z_t F(k_t, 1)$$

where u is the CRRA utility defined earlier in 3.10.
The recursive problem[11] of 3.16 using the value function in any period is then defined as:

$$V(k, z) = \max_{0 \le k' \le zF(k,1)+(1-\delta)k} u(z(F(k, 1) + (1-\delta)k - k') + \beta E[V(k', z')|z],$$

where the expectations are conditional on the realization of z. In the case of a Markov chain with m realizations $[z_1, ...z_m]$ and with the probability transition matrix $\Pi = [\Pi_{ij}]$, the expectation is given as

$$E[V(k', z')|z_i] = \sum_{j=1}^{m} \Pi_{ij} V(k', z_j).$$

In the case of a continuous valued Markov process with conditional probability function $\Pi(z, z')$ over the interval $[c, d]$, it is

$$E[V(k', z')|z] = \int_c^d V(k', z')\Pi(z, z')\, dz'.$$

The optimal capital path for any period is given by $k' = h(k, z)$ and as in the deterministic case we have to first find V. In this model, given $k_0 > 0$ we are looking for a sequence of random variables[12] $\{k_t\}_{t=1}^{\infty}$ that come about from the value function, V. The uniqueness and existence of V and h is developed in [10]. Unlike the deterministic model (where we find one equilibrium point for the capital), here we are interested in the limiting distribution of the sequence of random variables $\{k_t\}_{t=1}^{\infty}$. To show this type of weak convergence, we consider the following example:

Example 2 (The nature of the closed form solution in a model with aggregate shocks)**.** *Let $\delta = 1$ and $\eta = 1$, so that we have the log utility* [13] *below:*

$$u(c_t) = \ln(c_t).$$

Let $z_t \overset{iid}{\sim} G$ and assume that $k_{t+1} = h(k_t, z_t) = \alpha\beta z_t k_t^{\alpha}$.

[11]Notice the state space has now increased in dimension by the addition of aggregate shock with respect to the previous deterministic model.

[12]Or equivalently, a sequence of policy functions $\{h_t\}_{t=1}^{\infty}$.

[13]Using log utility here with $\delta = 1$ yields a closed form solution.

Proposition 6 $(k_{t+1} = h(k_t, z_t) = \alpha\beta z_t k_t^\alpha)$.

Proof: Reproduced from [4]: *Let the one-period utility function u and the production function with constant labor input (N=1) F, be given by:*
$u(c_t) = \ln(c_t),\ F(z_t, k_t, 1) = z_t F(k_t, 1) = z_t k_t^\alpha,\ \alpha \in (0,1)$ *and* $f(k_t) = z_t F(k_t) + (1-\delta)k_t$ *with* $\delta = 1$ *In the deterministic case, we found that* k_{t+1} *was directly proportional to* k_t^α. *So let's try the policy function as*

$$k_{t+1} = h(k_t, z_t) = A z_t k_t^\alpha$$

with the unknown parameter A. We use stochastic equivalent to the following identity (called the Euler equation):

$$\frac{u'(c_t)}{u'(c_{t+1})} - \beta F'(k_{t+1}) = 0$$

and substitute parameters of this model below

$$1 = \beta E_t \left[\frac{(1-A) z_t k_t^\alpha}{(1-A) z_{t+1} \left[A z_t k_t^\alpha \right]^\alpha} \alpha z_{t+1} \left[A z_t k_t^\alpha \right]^{(\alpha-1)} \right] = \frac{\alpha\beta}{A}.$$

If we set $A = \alpha\beta$, *the function* $h(k_t, z_t) = \alpha\beta z_t k_t^\alpha$ *indeed satisfies the above equation. Thus it is the policy function we are looking for.* ***Q.E.D.***

Given k_0 *and* $k_1 \sim \Psi_1$, *we obtain:*

$$\begin{aligned} \Psi_1(a) = Pr\{k_1 \le a\} &= Pr\{\alpha\beta z_0 k_0^\alpha \le a\} \\ &= Pr\{z_0 \le \frac{a}{\alpha\beta k_0^\alpha}\} \\ &= G\left(\frac{a}{\alpha\beta k_0^\alpha}\right). \end{aligned}$$

For successive periods we can define a transition function

$$H(a,b) = Pr\{k_{t+1} \le a | k_t \le b\} = G\left(\frac{a}{\alpha\beta k_t^\alpha}\right),\ \forall a, b > 0$$

$$\Psi_{t+1} = Pr\{k_{t+1} \le a\} = \int H(a,b)\ d\Psi_t(b),\ t = 0, 1, ..$$

where the distribution Ψ_0 *is simply a mass point for a given* k_0. *If h and G are some suitable family then H is such that* $\{k_t\}$ *converges to a unique limiting distribution satisfying*

$$\Psi(k') = \int H(k', k)\ d\Psi(k)$$

where Ψ *is called the invariant distribution and gives a probabilistic description of the asset* k_t *in any period t. It also describes the distribution of asset in periods* $t+1, t+2, \ldots$. *Generally we are interested in computing*

$$\int \phi(k)\ d\Psi(k) \tag{3.18}$$

for some continuous function ϕ. If we have the analytical form of Ψ, we could use importance sampling to obtain an estimate of the above integral, 3.18*. In our example, since* $k_{t+1} = \alpha\beta z_t k_t^{\alpha} \ \ \forall t, \ \forall\{z_t\}$ *then its logarithm is* $\ln(k_{t+1}) = \ln(\alpha\beta) + \alpha\ln(k_t) + \ln(z_t)$*. Since* $\{z_t\}$ *are iid then so are* $\{\ln z_t\}$*. Let* $\{lnz_t\} \overset{iid}{\sim} N(\mu, \sigma^2)$*. Given* k_0*,* $\{ln(k_t)\}_{t=1}^{\infty}$ *is a sequence of normally distributed random variables with means* $\{\mu_t\}_{t=1}^{\infty}$ *and variances* $\{\sigma_t^2\}_{t=1}^{\infty}$*. The limiting values of these means and variances can be computed as follows:*
Let μ_t *be the mean at time zero of the log of the capital stock in period* t*. Then,*

$$\begin{aligned}
\mu_t &= E_0[\ln k_t] \\
&= E_0[\ln(\alpha\beta) + \alpha\ln(k_{t-1}) + \ln(z_{t-1})] \\
&= \ln(\alpha\beta) + \mu + \alpha\mu_{t-1} \\
&= \ln(\alpha\beta) + \mu + \alpha[\ln(\alpha\beta) + \mu] + \alpha^2\mu_{t-1} \\
&= [\ln(\alpha\beta) + \mu][1 + \alpha + ... + \alpha^{t-1}] + \alpha^t\mu_0 \\
&= [\ln(\alpha\beta) + \mu]\left[\frac{\alpha^t - 1}{\alpha - 1}\right] + \alpha^t\mu_0 \\
&= \left[\mu_0 - \frac{\ln(\alpha\beta) + \mu}{1 - \alpha}\right]\alpha^t + \frac{\ln(\alpha\beta) + \mu}{1 - \alpha}
\end{aligned}$$

where the expectation is conditioned on information at time $t = 0$*.*
Since $0 < \alpha < 1$*, then*

$$\lim_{t\to\infty} \mu_t = \frac{\ln(\alpha\beta) + \mu}{1 - \alpha}.$$

If we let

$$\mu^* = \lim_{t\to\infty} \mu_t$$

and with $\mu = 0$*, we obtain*

$$e^{\mu^*} = (\alpha\beta)^{\frac{1}{1-\alpha}}.$$

which is a similar to the solution, k^**, in example 3. Similarly, define* σ_t^2 *as the variance at time zero of the logarithm of the capital stock in period* t*. Then*

$$\begin{aligned}
\sigma_t^2 &= Var_0[\ln k_t] \\
&= Var_0[\ln(\alpha\beta) + \alpha\ln(k_{t-1}) + \ln(z_{t-1})] \\
&= \alpha^2\sigma_{t-1}^2 + \sigma^2
\end{aligned}$$

is an ordinary differential equation with a solution given by

$$\sigma_t^2 = \left[\sigma_0^2 - \frac{\sigma^2}{1 - \sigma^2}\right]\alpha^{2t} + \frac{\sigma^2}{1 - \alpha^2}.$$

Since $0 < \alpha < 1$*,*

$$\lim_{t\to\infty} \sigma_t^2 = \frac{\sigma^2}{1 - \alpha^2}.$$

Here the invariant distribution function of $\{ln(k_{t+1})\}_{t=0}^{\infty}$ *is* $\Psi \sim N\left(\frac{\ln(\alpha\beta)+\mu}{1-\alpha}, \frac{\sigma^2}{1-\alpha^2}\right)$*. Most of these models do not have such nice* h *and* G *functions and hand calculations are not possible. In fact only a handful of examples have closed form solutions, and we must have numerical solutions.*

We will now look at an alternative numerical example.

Example 3 (The nature of the non-analytic solution in a model with aggregate uncertainty)**.** *Consider the same farmer's expected utility maximization problem but with calibrations that only leads to a numerical solution.*

$$\max_{c_t, k_{t+1}} \; E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \quad \beta \in (0,1)$$

where

$$u(c_t) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0$$

and such that

$$\begin{aligned} k_{t+1} + c_t &\leq z_t k_t^{\alpha} + (1-\delta)k_t, \quad \alpha \in (0,1) \\ 0 &\leq c_t, \\ 0 &\leq k_{t+1} \end{aligned}$$

with k_0, z_0 given and $t = 0, 1, \ldots$. Assume the Markov chain for the aggregate shock z_t has three states, $z = [0.99, 1.00, 1.01]$ and the transition matrix is given by

$$\Pi = \begin{bmatrix} 0.61 & 0.34 & 0.05 \\ 0.24 & 0.52 & 0.24 \\ 0.05 & 0.34 & 0.61 \end{bmatrix}.$$

Choose the parameters $\alpha = 0.27, \beta = 0.984, \delta = 0.01,$ and $\eta = 2$ so there is no closed form solution.

Algorithm 2 (Computation of Equilibria in a Aggregate Shock Economy for example 3)**.**

1. *Choose*[14] *a grid of n equally spaced points over $[k_{min}, k_{max}] = [0.1, 1.5]$.*
2. *Initialize*[15] *the value function to $V^0 = u^T(I - \beta\Pi)^{-1}$.*
3. *Compute the new value function by*

$$V_{ij}^{s+1} = \max_{k_d} \; u(z_j F(k_i, 1) + (1-\delta)k_i - k_d) + \beta \sum_{t=1}^{\infty} \Pi_{jl} V_{dl}^{s}$$

   for $i, d = 1, ..., n$ and $j = 1, ..., m$. Repeat until V converges.

In the light of the previous example and the nature of the closed form solution, we would expect the equilibrium points to be a function of the limiting probability distribution of the

[14] The grid should contain the equilibrium points and is chosen by having a-priori knowledge of the problem or by trial and error.

[15] This initialization gives us an estimate of the stationary elements of V. In addition, to evaluate $u(c_t)$, set $c_t = z_t k_t^{\alpha} + (1-\delta)k_t - k_{t+1}$.

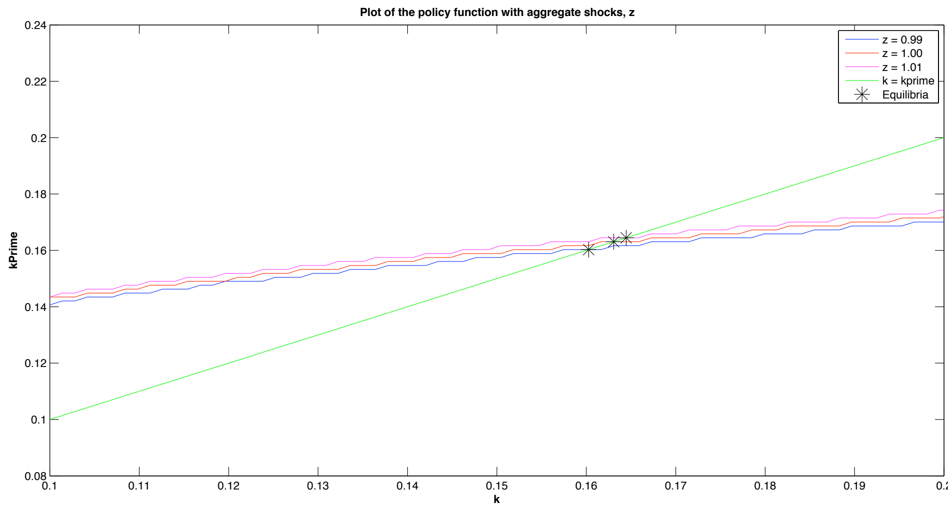


Figure 3.3: Policy function with Equilibria in the Aggregate Shock Economy for example 3

random variable, z. This would be a discrete probability distribution that should give us an equivalent number of discrete equilibrium points. In this case, k_t *converges to three equilibrium points, i.e.* $\{k_\infty\} \in$ *[0.16026, 0.16306, 0.16446]. These points are displayed in the above Figure 3.3, and as expected these are the points of intersection between the respective policy curve and the* 45° *line. Please note, since we zoom in to focus on the equilibrium points which are at the intersection of the policy curves and the* 45° *line, wiggles appear as an artifact of the grid size. The policy curve should be smooth at least for this example!*
The last two examples exemplify that the limiting distribution of aggregate shocks characterizes the limiting sequence of random variable $\{k_t\}_{t=1}^{\infty}$*. We hope to apply a similar reasoning in the models presented in the next chapter.*

CHAPTER 4

MODEL WITH IDIOSYNCRATIC UNCERTAINTY

In the previous models, we considered a single farmer (or household) acting as a producer and consumer. In the next two models, there is a single consumer per household. Consumers, agents and households are used interchangeably from now on. Furthermore, models will have three sectors: a continuum of households who maximize consumption and savings, a single firm - owned collectively by these households - that maximizes profits and will be responsible for production, and a government who redistributes wealth by charging tax and paying unemployment benefits. This type of model is called a heterogeneous agent based economy where households differ in wealth and employment status (unlike the homogeneous agent based economy discussed earlier where a single entity represented the entire economy's behavior). The lucky employed ones store wealth and run it down during periods of unemployment. These models are of prime interest to our work as the solution is challenging and computationally expensive. We have seen the implications of aggregate uncertainty from the previous model; in this chapter we want to consider the effect of employment uncertainty (idiosyncratic uncertainty) on the solution. In the next chapter, we will consider both uncertainties simultaneously.
We will use uppercase letters to denote aggregate parameters (parameters relating to the entire economy), and lowercase letters for parameters relating to households. For instance c_t will be a single household consumption for period t, while C_t represents the consumption of all households collectively (an aggregate variable). There is one exception in the notation: the individual wealth in period t will be denoted by a_t and aggregate wealth K_t. This is because the aggregate wealth is assumed to be held (borrowed) by the firm, and the firm and households solve distinct maximization problems in each time period.
We present the sectors in the current economy - Households, Government and Firms:

4.1 Households

The agent is faced with uninsurable uncertainty of employment at the beginning of any period. This introduces heterogeneity at the household level, and is the only source of randomness in the economy. The household employment state follows a first order Markov

Chain:

$$\Pi(\epsilon'|\epsilon) = Pr\{\epsilon_{t+1} = \epsilon'|\epsilon_t = \epsilon\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix}, \tag{4.1}$$

where p_{ue} is the probability of being employed in the next period while being unemployed in the current period.

The Household Problem. At the beginning of any period, the agent learns about his/her employment status.[1] With known (r_t, w_t) and the wealth at the end of the last period, the agent decides on consumption for the current period and savings for the beginning of the next period.

All households have the same utility (CRRA) function $u(c)$ as in the deterministic case and face the same problem, so we analyze a single consumer's decision problem. For the ease of reference, the CRRA utility function is restated below:

$$u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta} & \text{if } 0 < \eta < 1 \ \& \ \eta > 1 \\ \ln(c) & \text{if } \eta = 1 \end{cases} \tag{4.2}$$

As before each household maximizes its lifetime utility:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \tag{4.3}$$

where $\beta \in (0,1)$ is the constant discount factor, and expectations are conditioned on the information at time zero.

The following constraint exists on the households in period t:

$$a_{t+1} = \begin{cases} (1 + (1-\tau_t)r_t)a_t + (1-\tau_t)w_t - c_t, & \text{if } \epsilon = e \\ (1 + (1-\tau_t)r_t)a_t + m_t - c_t, & \text{if } \epsilon = u \end{cases} \tag{4.4}$$

where a_t is the end of the $t-1$ period household wealth, c_t is the consumption in period t, τ_t is the income tax rate in period t, r_t is the interest rate for period t, and m_t and w_t are the unemployment benefits and wages for period t, respectively. The after tax post interest savings wealth set aside from the previous period is given as $(1 + (1-\tau_t)r_t)a_t$. Likewise the post tax wages employed households earn are given as $(1-\tau_t)w_t$, while the unemployed agents are endowed with tax-free[2] unemployment compensation m_t in period t.

4.2 The Firm

There is a single firm owned collectively by all households.[3] It is the sole producer in this economy. The production function is defined as:

$$F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}, \ \ \alpha \in (0,1) \tag{4.5}$$

[1]In this paper, the employment status is an exogenous variable or randomly ordained. Costain(1997), Den Haan et al (2000), and Heer (2003) are few studies where the employment status is updated due to the model friction (or endogenously).

[2]This is not true for the US unemployment policy i.e. unemployment benefits are taxable.

[3]If the firms are owned by another entity there would be competitive optimizing as in game theory. The famous Nash equilibrium introduces a potential solution to multiparty optimality.

where K_t, and N_t are the aggregate (or total) capital and the total percentage of employed agents in the economy in period t, respectively. The firms takes the aggregate capital K_t and labor N_t from the households, produces and pays out interest for the borrowed capital and wages for labor. It is assumed that the total capital across households is taken by the firm for production. There is no alternative source of interest income such as a bank.
The firm then maximizes profits with respect to labor and capital. It has to decide on the amount of (from the households) aggregate capital, K_{t+1} to set aside for the next period $t+1$, and aggregate labor N_t to rent for current period t. This optimization is based on the current interest rate, r_t and the wage rate, w_t. The firms maximizes profits based on:
Firm's Decision problem:

$$\max_{(K_{t+1},N_t)} Y_t - r_t K_{t+1} - w_t N_t, \tag{4.6}$$

with

$$Y_t = F(K_t, N_t) + (1-\delta)K_t$$

The total resource constraint of any t period is given by:

$$C_t + K_{t+1} \leq F(K_t, N_t) + (1-\delta)K_t. \tag{4.7}$$

That is:
Total consumption for period t + *Total savings for period* $t+1 \leq$ *Total production for period* t + *Depreciated total capital from period* $t-1$.

The interest rate and the wages for period t are set using the firm's first order conditions:

$$r_t \quad = \tfrac{\partial}{\partial K_t}[F(K_t, N_t) + (1-\delta)K_t] \quad = \alpha \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta \tag{4.8}$$

$$w_t \quad = \tfrac{\partial}{\partial N_t}[F(K_t, N_t) + (1-\delta)K_t] \quad = (1-\alpha) \left(\frac{K_t}{N_t}\right)^{\alpha} \tag{4.9}$$

where $\delta \in [0,1]$ denotes the depreciation rate of capital. Since the households collectively determine how much capital K_{t+1} and labor N_t, the firm's problem becomes a static one. Hence our main focus remains on the household decision problem.

4.3 Government

The sole purpose of the government is to redistribute capital. It performs this by charging tax, τ_t, and paying unemployment benefits, m_t to households. The government balances its budget in every period, i.e.

$$T_t = M_t$$

where T_t is the aggregate (total) tax revenue received and M_t is the total unemployment benefits paid out to unemployed households in period t. Thus the government decides how much unemployment benefits, m_t, need to be paid out based on the total unemployment and then determines the tax rate, τ_t. As with the firm, the government's problem is also a static one.

4.4 In Course to the Stationary Equilibrium

At the start of period t, the households learn about their employment status, the interest and the wage rates. Since the firm and the government's problems have implied solutions through the household optimization, solving the household problem is imperative.

4.4.1 The Household Problem- Restated

The household has to maximize its sequential utility 4.3 which can be restated as the following recursive[4] dynamic programming problem along with the employment transition matrix 4.1 and subject to household constraint equations 4.4:

$$V(\epsilon_t, a_t) = \max_{c_t, a_{t+1}} [u(c_t) + \beta E\{V(\epsilon_{t+1}, a_{t+1}) | \epsilon_t\}] \tag{4.10}$$

where the period invariant individual state space consists of sets $(\epsilon, a) \in \chi = \{e, u\} \times [a_{min}, \infty)$. Note in the above equation, $a_{t+1} = a_{t+1}(\epsilon_t, a_t)$. The agent has to decide how much to consume in the current period and how much to save for the next period based on the current interest and wage rates. The unemployment benefits rate, m_t, is set[5] such that households will not substitute unemployment for work. The order in which decisions are made for each entity is presented in the decision process model diagram Figure 4.1 on the next page.

4.5 The Stationary Equilibrium

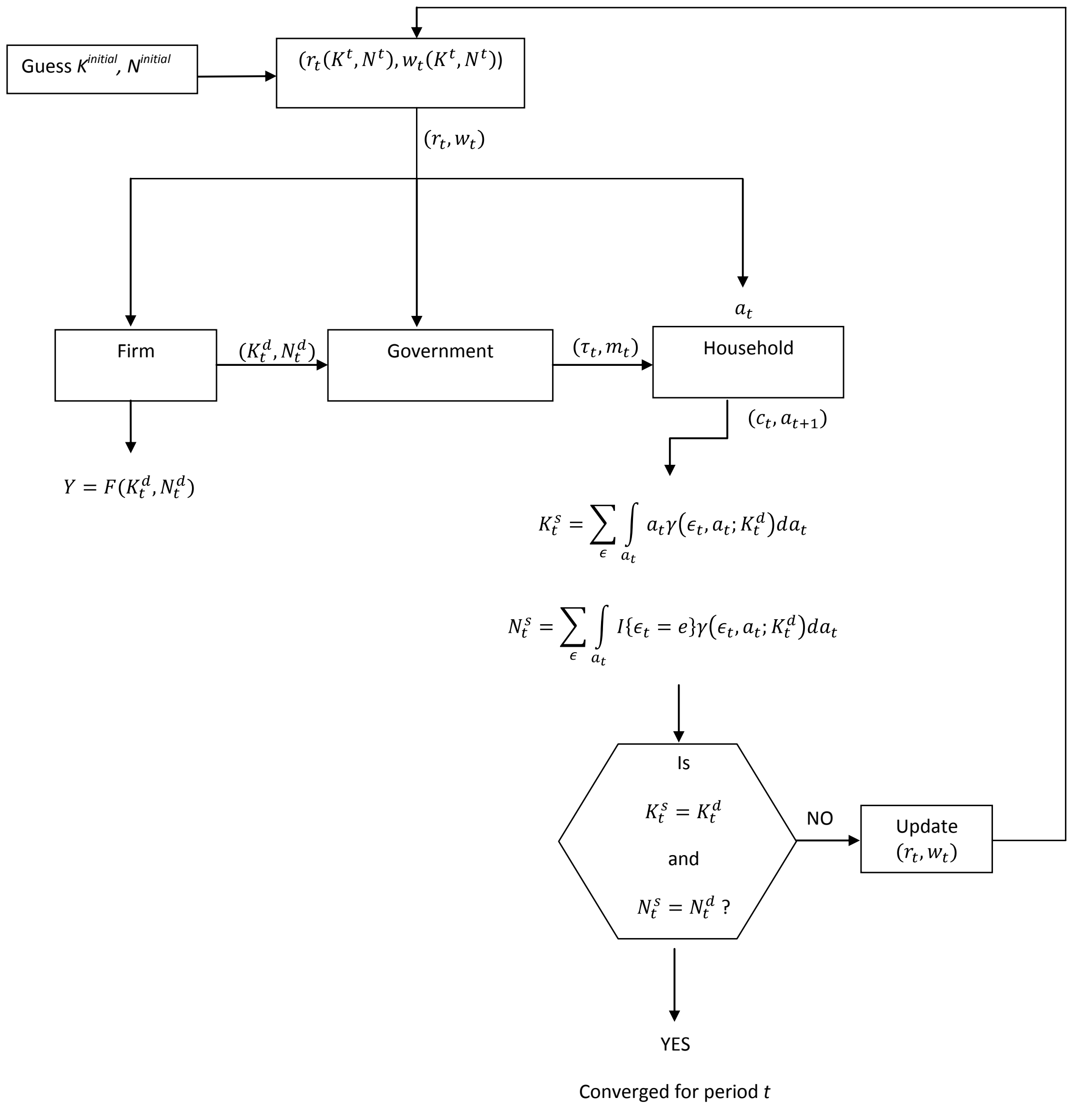
As mentioned before, the only source of randomness comes from employment uncertainty that follows a stochastic matrix, Π. This matrix gives us the invariant probability distribution of employment and unemployment. With these parameters, we are interested in computing the wealth distributions of the households where the distribution of assets and the number of employed and unemployed agents are constant; but, individual agents are not characterized by constant wealth and employment status.
At stationary equilibrium, the aggregate variables and factor prices are constant and we drop the time indices and denote next period values using the 'prime' notation, $'$, as necessary.

Conditions for the stationary equilibrium. For a given set of factor prices (r, w) and government policy (m, τ), the stationary equilibrium is: a value function $V(\epsilon, a)$ and a sequence of individual decision rules $a' = a'(\epsilon, a)$ and $c(\epsilon, a)$ that solves the household optimization problem 4.3, and a stationary probability distribution function $\Gamma(\epsilon, a)$ or equivalently an invariant density function $\gamma(\epsilon, a)$ for household wealth. In addition to these, other conditions have to be satisfied. A complete summary of the conditions to be satisfied simultaneously is given below.

[4][10] justifies this sequential to recursive formulation. This justification is similar to the deterministic one shown in the previous chapter.

[5]In the literature the ratio of unemployment compensation to net wage income is called the replacement ratio and is typically set to 25%.

At the beginning of period t, ϵ_t is realized.



Decision Process for Each Entity in the Economy

Model in Section 4 – Economy with idiosyncratic uncertainty

Figure 4.1: Decision process and arguments of each entity in a economy with Idiosyncratic uncertainty

a) Households are uniformly distributed with measure one. The individual household maximizes

$$V(\epsilon, a) = \max_{c,a'} \left[u(c) + \beta E\{V(\epsilon', a'\}|\epsilon \right],$$

s.t.

$$a' = \begin{cases} (1 + (1-\tau)r)a + (1-\tau)w - c, & \text{if } \epsilon = e \\ (1 + (1-\tau)r)a + m - c, & \text{if } \epsilon = u \end{cases}$$

$$a \geq a_{min}$$

$$\Pi(\epsilon'|\epsilon) = Pr\{\epsilon_{t+1} = \epsilon'|\epsilon_t = \epsilon\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix}$$

b) The distribution of (ϵ, a) is stationary. The aggregate capital K, aggregate consumption C, and aggregate employment N are constant.

c) Factor prices are equal to their respective marginal products:

$$\begin{aligned} r &= \alpha \left(\frac{N}{K} \right)^{1-\alpha} - \delta \\ w &= (1-\alpha) \left(\frac{K}{N} \right)^{\alpha}. \end{aligned}$$

d) The government budget balances: $M = T$.

e) The aggregate consistency conditions hold:

$$\begin{aligned} K &= \sum_{\epsilon \in \{e,u\}} \int_{a_{min}}^{\infty} a\, \gamma(\epsilon, a) \; da, \\ N &= \int_{a_{min}}^{\infty} \gamma(\epsilon = e, a) \; da, \\ C &= \sum_{\epsilon \in \{e,u\}} \int_{a_{min}}^{\infty} c(\epsilon, a)\, \gamma(\epsilon, a) \; da, \\ T &= \tau(wN + rK), \\ M &= (1-N)m, \end{aligned}$$

and finally,

f) The goods market clears:

$$C + K' = F(K, N) + (1-\delta)K.$$

Recall that K and N are the aggregate capital and employment, respectively, C is the aggregate consumption for the household, and T and M are the aggregate tax received and unemployment benefits paid out by the government, respectively.

The following algorithm describes a general approach to solve the household problem.

Algorithm 3. *: **A general algorithm to compute the Stationary Equilibrium***

Step 1: Compute the stationary employment N using normalized eigenvectors[6] *or an alternative technique.*

Step 2: Make initial guesses of the aggregate capital stock K and the tax rate τ.

Step 3: Compute the wage rate w and the interest rate r.

Step 4: Compute the household's decision rules, $\{a_{t+1}\}_{t=0}^{\infty}$.

Step 5: Compute the invariant density[7] *of assets for the employed and unemployed agents.*

Step 6: Compute the capital stock K and taxes T that solve the aggregate consistency conditions.

Step 7: Compute the tax rate τ that solves the government budget.

Step 8: Update K and τ and return to step 3 if necessary.

Since we are familiar with the computation of the value function, we focus on step 5 - computing the invariant density below. The main idea in all these techniques is to set a grid on household assets, and compute all optimal transitions for each grid point to determine the household policy.

4.6 Existing Algorithms to Compute the Stationary Density

Four standard techniques discussed in [4] are given here.

Method 1: Computation of the Stationary Distribution Function, $\Gamma(\epsilon, a)$ by discretization of the distribution function.

Method 2: Computation of the Stationary Density Function, $\gamma(\epsilon, a)$ by discretization of the density function.

Method 3: Computation of the Stationary Distribution Function $\Gamma(\epsilon, a)$ by Monte Carlo Simulation.

Method 4: Computation of the Stationary Distribution Function $\Gamma(\epsilon, a)$ by Functional Approximation.

Algorithm 4 (Method 1: Computation of the stationary distribution by discretization)**.**

Step 1 : Place a grid on the asset space $A = \{a_1 = a_{min}, a_2, ..., a_m = a_{max}\}$ such that the grid is finer than the one used to compute the optimal decision rules a'

[6] Since Π is a markov matrix, we normalize the eigenvector associated with eigenvalue of 1 to get the employment probability density.

[7] Alternatively, you could compute the invariant cumulative distribution function.

Step 2 : Choose an initial piecewise distribution function $\Gamma_0(e,a)$ *and* $\Gamma_0(u,a)$ *over the grid. Each will have m rows.*

Step 3 : Compute the inverse of the decision rule $a'(\epsilon,a)$

Step 4 : Iterate on

$$\Gamma_{i+1}(\epsilon',a') = \sum_{\epsilon=e,u} \pi(\epsilon',\epsilon)\Gamma_i\left(a'^{-1}(\epsilon,a'),\epsilon\right)$$

on the grid points (ϵ',a')

Step 5 : Iterate until Γ *converges*

Brief Description:

Step 1: This step is required for all algorithms. The choice of grid size depends on the calibration of parameters. A rough estimate would be to set a wide interval centered at the aggregate capital stock, K, of the representative agent economy (where there is no uncertainty). If during the first few runs a large proportion of households have wealth at the the end points of the grid, then increase the grid size. Thus setting the grid size requires a lot of trial and error. In our technique, we will overcome this by taking a very wide grid and using the bisection method. This technique is robust and does not require frequent updates on the grid.

Step 2: Unfortunately [4] reports that the choice of the initial distribution function and the number of simulations influence the convergence of this algorithm for the example they use. That is, if the distribution is initialized to

$$\Gamma_0(\epsilon,a) = \frac{a-a_{min}}{a_{max}-a_{min}},$$

instead of

$$\Gamma_0(\epsilon,a) = \begin{cases} 1, & \text{if } a \geq K \\ 0, & \text{if } else \end{cases}$$

the algorithm fails. Alternatively, if the number of simulations over the distribution are increased from $\{500, 1000, 1500, ..., 25000\}$ to $\{2500, 5000, ..., 125000\}$, the algorithm fails. This was a little alarming as it shows the fragility of this algorithm. In principle, the initial distribution should not influence convergence. This is because the stationary density should be independent of the starting states. The number of simulations should be sufficiently large at every step to guarantee convergence to the stationary distribution. We address these concerns and cite the source of failure - the termination condition and the update rule on the aggregate capital stock, K. These are not mentioned in the algorithm, but were found in the actual gauss programs provided by [4]. In the program, initial interest and wage rates are updated using K. And K is updated as follows:

$$K_{next} = \psi K_{previous} + (1-\psi)K_{computed} \tag{4.11}$$

where

$$\begin{aligned} K_{computed} &= \sum_{\epsilon \in \{e,u\}} \int_{a_{min}}^{\infty} a\gamma(\epsilon, a)\, da \\ &\approx \sum_{\epsilon} \left(\sum_{j=2}^{m} (\Gamma(\epsilon, a_j) - \Gamma(\epsilon, a_{j-1})) \frac{a_j + a_{j-1}}{2} + \Gamma(\epsilon, a_1) a_1 \right). \end{aligned}$$

ψ is what I call a parameter that defines the rate of convergence of K. The value chosen for ψ was 0.95. The choice of this parameter is the root of failure. We explain this by setting a grid on K_{in} and determining K_{out}. Figure 4.3 shows a nearly vertical relationship that would be very sensitive to convergence, and failures are imperative using the update equations above.

Step 3: The inverse of the policy, $a'(\epsilon, a)$, where $a = a'^{-1}(\epsilon, a_j), j = 1, ..., m$ is computed over the chosen grid. Linear interpolation is used for the computation of $a'(\epsilon, a)$ for $a_j < a < a_{j+1}$, and again to find the inverse. [5] establishes that a' is strictly nondecreasing in a.

Step 4: The distribution function is updated in a straight forward manner.

Algorithm 5 (Method 2: Computation of the stationary density function by discretization)**.**

Step 1: Place a grid on the asset space $\mathcal{A} = \{a_1 = a_{min}, a_2, ..., a_m = a_{max}\}$ *such that the grid is finer than the one used to compute the optimal decision rules following [9].*

Step 2: Set $i = 0$*. Choose initial discrete density functions* $\gamma_0(\epsilon = e, a)$ *and* $\gamma_0(\epsilon = u, a)$ *over that grid. The two vectors have* m *rows each, where* m *is the size of the finer grid set in the last step.*

Step 3: Set $\gamma_{i+1}(\epsilon, a) = 0$ *for all* ϵ *and* a*. i) For every* $a \in \mathcal{A}, \epsilon \in \{e, u\}$*, compute the optimal next-period wealth* $a_{j-1} \leq a' = a'(\epsilon, a)$ *and ii) for all* $a' \in \mathcal{A}$ *and* $\epsilon' \in \{e, u\}$ *the following sums:*

$$\begin{aligned} \gamma_{i+1}(\epsilon', a_{j-1}) &= \sum_{\epsilon=e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1} \leq a'(\epsilon,a) < a_j}} \Pi(\epsilon'|\epsilon) \frac{a_j - a'}{a_j - a_{j-1}} \gamma_i(\epsilon, a), \\ \gamma_{i+1}(\epsilon', a_j) &= \sum_{\epsilon=e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1} \leq a'(\epsilon,a) < a_j}} \Pi(\epsilon'|\epsilon) \frac{a' - a_{j-1}}{a_j - a_{j-1}} \gamma_i(\epsilon, a). \end{aligned}$$

Step 4: Iterate until γ *converges.*

Brief Description: The steps are essentially the same as in algorithm 4 except the density function is now updated. If the optimal next period capital stock a' is $\in (a_{j-1}, a_j)$, we set

$$a'_{new} = \begin{cases} a_j & \text{wp } \frac{a' - a_{j-1}}{a_j - a_{j-1}} \\ a_{j-1} & \text{wp } 1 - \frac{a' - a_{j-1}}{a_j - a_{j-1}} \end{cases}$$

The reason why $a_{j-1} < a' < a_j$ is because the density grid is finer than the grid on which the optimal policy is computed using dynamic programming. Overall this algorithm is slightly faster with respect to algorithm 4 as we are not computing the inverse of the policy, $a'(\epsilon, a)$.

Algorithm 6 (Computation of the Stationary Distribution Function $\Gamma(\epsilon, a)$ by Monte Carlo Simulation)**.**

Step 1: Choose a large sample of households, N $(= 1000)$

Step 2: Initialize the sample. Each household $i = 1, ..., N$ is assigned an initial wealth a_0^i and employment status ϵ_0^i

Step 3: Compute the next period wealth level $a'(\epsilon^i, a^i)$ $\forall$ $i = 1, ..., N$

Step 4: Use a random number generator to obtain ϵ'^i $\forall$ $i = 1, ..., N$

Step 5: Compute a set of statistics - mean and standard deviation - from this sample

Step 6: Iterate until the distributional statistics converge

Algorithm 7 (Computation of the stationary distribution using Functional Approximation)**.**

Step 1: : Choose initial moments μ^* and $(\sigma^\epsilon)^2$ for the wealth distribution $\epsilon \in \{u, e\}$ and compute the corresponding parameters ρ^ϵ of the exponential distribution

Step 2: : Compute the moments of the next-period wealth distribution for the employed and unemployed agents; if $\epsilon = e$,

$$
\begin{aligned}
\mu^{\epsilon'} &= \pi(e|e)\rho_0^e \int_{-\infty}^{a_{max}} \max(a'(e,a), a_{min}) e^{\rho_1^e a + \rho_2^e a^2} \, da \\
&+ \pi(e|u)\rho_0^u \int_{-\infty}^{a_{max}} \max(a'(u,a), a_{min}) e^{\rho_1^u a + \rho_2^u a^2} \, da \\
(\sigma^{\epsilon'})^2 &= \pi(e|e)\rho_0^e \int_{-\infty}^{a_{max}} (\max(a'(e,a), a_{min}) - \mu^e)^2 e^{\rho_1^e a + \rho_2^e a^2} \, da \\
&+ \pi(e|u)\rho_0^u \int_{-\infty}^{a_{max}} (\max(a'(u,a), a_{min}) - \mu^u)^2 e^{\rho_1^u a + \rho_2^u a^2} \, da
\end{aligned}
$$

Step 3 : Iterate until the moments μ^ϵ and σ^ϵ converge

Example 4. *Let the agent utility function be given by*

$$u(c_t) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0$$

with $\eta = 2.0$

$$\Pi(\epsilon'|\epsilon) = \begin{pmatrix} 0.500 & 0.500 \\ 0.0435 & 0.9565 \end{pmatrix}. \tag{4.12}$$

The firm's production function as described in 4.5 with $\alpha = 0.36$*, i.e.*

$$F(K_t, N_t) = K_t^{0.36} N_t^{1-0.36}.$$

Let the individual discounting factor, β*, and capital depreciation rate,* δ*, be* 0.995 *and* 1*, respectively.*

The stationary density plots using all four algorithms in 4.6 are given in Figure 4.2.

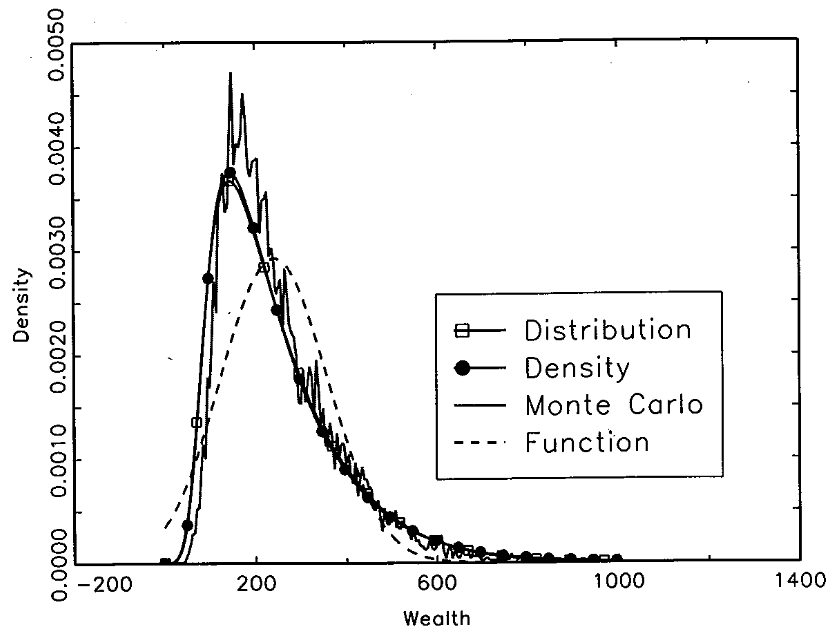


Figure 4.2: Stationary density plots from [4]

Table 4.1: Running Time Analysis for Algorithms by [4]

	Stat. Dist.	Stat. Dens.	Monte Carlo	Func Approx
Mean	243.7	243.7	243.4	246.6
Running Time (hrs.)	28.25	20.16	75.51	19.05

For example 4, the programs were coded in GAUSS and run on a Pentium III with 846 MHz. As expected, the plots using the first two algorithms - Density and Distribution Functions - overlap each other. The Monte Carlo method could produce a smoother density if the number of households is increased. The number of households for this run was 1000. If a larger number of households are used say 5000 or 10000, the running time increases to days. Also note in table 4.6, Monte Carlo simulation takes the longest time to converge. For the Functional Approximation method, the stationary density is more symmetric and has a smaller variance compared to the other algorithms. For this reason, [4] cites skepticism in using this technique. The next best technique in terms of running time is Method 2 - discretization of density function - that takes 20.16 hours. We will compare our method directly to this algorithm is running time and convergence.

4.6.1 Our method

Algorithm 8. *: A general algorithm to compute the Stationary Equilibrium using a grid on aggregate capital stock*

Step 1: *Set a grid on the aggregate capital stock, KGrid.*

Step 2: *Compute the stationary employment N using normalized eigenvectors*[8] *or an alternative technique.*

Step 3: *For each aggregate capital stock value in KGrid, call this K_{in}, compute the following:*

Step 3a: *Compute the wage rate w tax rate τ, and the interest rate r.*

Step 3b: *Compute the household's decision rules, $\{a_{t+1}\}_{t=0}^{\infty}$, using dynamic programming (value function)*

Step 3c: *Compute the invariant density of assets for the employed and unemployed agents using the Perron-Frobenius Theorem.*

Step 3d: *Compute the mean of the above density. Call this aggregate capital stock $K_{computed}$.*

Step 3e: *Compute other aggregate consistency parameters: T, K, C, and M.*

Step 3f: *Compute the tax rate τ that solves the government budget.*

Step 4: *Stop if $|K_{in} - K_{computed}| < \epsilon$*

We set a coarse grid on the aggregate capital stock and look for the value of $K_{computed}$ that is approximately same as K_{in} value. At equilibrium these two values should be the same. Among the four methods to compute the stationary density discussed above, the algorithm that discretizes the stationary density seems fastest[9]. If this method is used to compute the density for each value of the aggregate grid, the running time would be days because we would have to run large number of iterations get stationary density. To overcome this, we require a technique that is relatively fast in computing the stationary density for each K_{in} value. We propose using the Perron-Frobenius Theorem.

Theorem 4 (Peron-Frobenius Theorem). *If $\Pi^n >> 0$ for some non-negative integer n, then $\exists$ an $X >> 0$ such that $X\Pi = X$, and if λ is any other eigenvalue of Π, then $|\lambda| < 1$*

Although the condition required in the Peron-Frobenius theorem is easy to state, it is difficult to establish for a given Π. To satisfy this condition, we use the following theorem.

[8] Since Π is a markov matrix, we normalize the eigenvector associated with eigenvalue of 1 to get the employment probability density.

[9] Note that the author lures to skepticism in using the functional approximation which is faster than the discretization method.

Theorem 5. *An irreducible, aperiodic, homogeneous Markov Chain on a finite state space has the property,* $\Pi^n >> 0$ *for some* $n > 0$*. Furthermore, this Markov chain has a unique probability distribution* P *such that* $P\Pi = P$*.*

Once a Markov matrix satisfying the properties of Theorem 5 is found, the stationary probability can be obtained by normalizing X into a probability vector, P. We construct a $2 * aGrid \times 2 * aGrid$ $a - a'$ individual policy transition matrix A from the optimal policy derived using the value function iteration. The transition matrix $\in \mathbb{R}^{2m} \times \mathbb{R}^{2m}$ for (a, ϵ) is constructed as follows:

$$\begin{aligned} P(a_{t+1}, \epsilon_{t+1} | (a_t, \epsilon_t)) &= P(a_{t+1} | \epsilon_{t+1}, (a_t, \epsilon_t)) P(\epsilon_{t+1} | (a_t, \epsilon_t)) \\ & \quad a_{t+1} \text{ is independent of } \epsilon_{t+1} \\ &= P(a_{t+1} | (a_t, \epsilon_t)) P(\epsilon_{t+1} | (a_t, \epsilon_t)) \\ &= P(a_{t+1} | (a_t, \epsilon_t)) \Pi(\epsilon_{t+1} | \epsilon_t). \end{aligned}$$

The matrix $P(a_{t+1}|(a_t, \epsilon_t))$ on the last line is the result of the value function iteration. The second matrix $\Pi(\epsilon_{t+1}|\epsilon_t)$ is set up such that the long term employment and unemployment match country specific economies. As mentioned before, this *aGrid* on which the density is computed is finer than the one used to compute the optimal decision rules following [9]. One way to set up the matrix A is as follows: if a' computed from the value function is between a_j and a_{j-1} on the finer grid, we set the new a' to be a_j with probability $\frac{a' - a_{j-1}}{a_j - a_{j-1}}$ and a_{j-1} with probability $1 - \frac{a' - a_{j-1}}{a_j - a_{j-1}}$. The matrix A is sparse in nature.[10] To satisfy the irreducible and aperiodic conditions given in Theorem 5, we perturb the matrix by adding a constant ϵ to all zero entries. The value of ϵ will influence the eigenvectors if the entries prior to perturbation are small. To resolve this, we set $\epsilon = \frac{minimum(A)}{2 * length(aGrid)}$. To guarantee reproducibility of results, we recommend ϵ be a constant value rather than a function of $minimum(A)$. For the examples in this paper $\epsilon = 10^{(-10)}$ works well. Needless to say this constant is smaller than $\frac{minimum(A)}{2 * length(aGrid)}$. Once we replace the zero entries of the matrix with ϵ, we simply use eigen-analysis to find the unique eigenvector corresponding to the eigenvalue equal to 1. The existence such a vector is guaranteed by theorem 4. We normalize this eigenvector to obtain the unique probability density mentioned in 5.

Comparative Analysis of Methods. We solve the model given in example 4 using our method described in algorithm 8. Figure 4.3 below plots the $K_{in} \in kGrid$ and K_{out} values. Note the slope of the curve is almost undefined. This explains the failures in algorithm 4 and 5. The update function 4.11 would only work if K_{next} chosen using ψ is very close to the K^* value in our plot. Their choice of ψ is a long process of trial and error, and the value chosen will not be uniform across all examples. Figure 4.4 is a close up view of the plot to approximate K^*. The point of intersection between the 45° line and the curve is the equilibrium aggregate stock value, K^*, approximately 242.54. This K^* is close but not the same as reported in table 4.6. As expected, if we use the K^* as the input value reported in table 4.6, we do not get it back in $K_{computed}$ using algorithms in [4]. One possible source of this error in algorithm 5 (or 4) is that the value function

[10][12] uses a similar matrix and maps the normalized eigenvector to a point in the probability function space using a min-max operator.

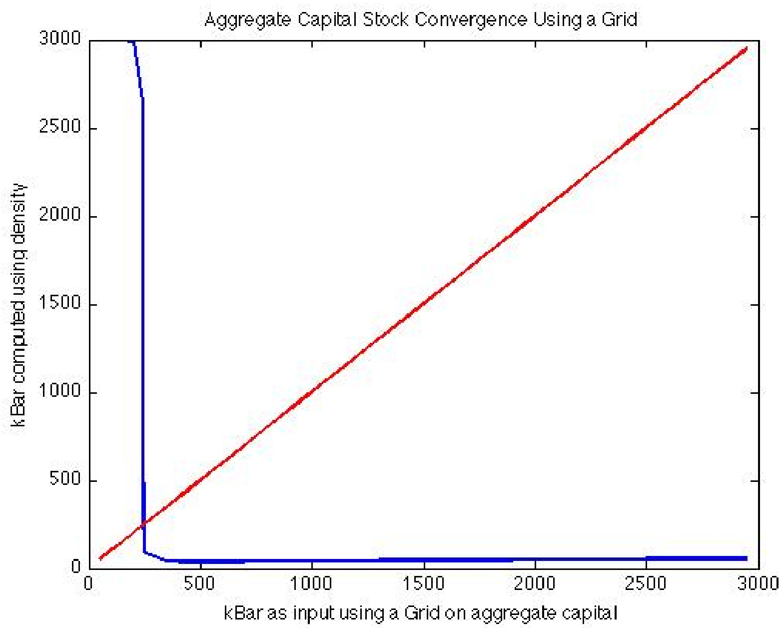


Figure 4.3: Aggregate Capital Stock at Equilibrium Using a Grid Search

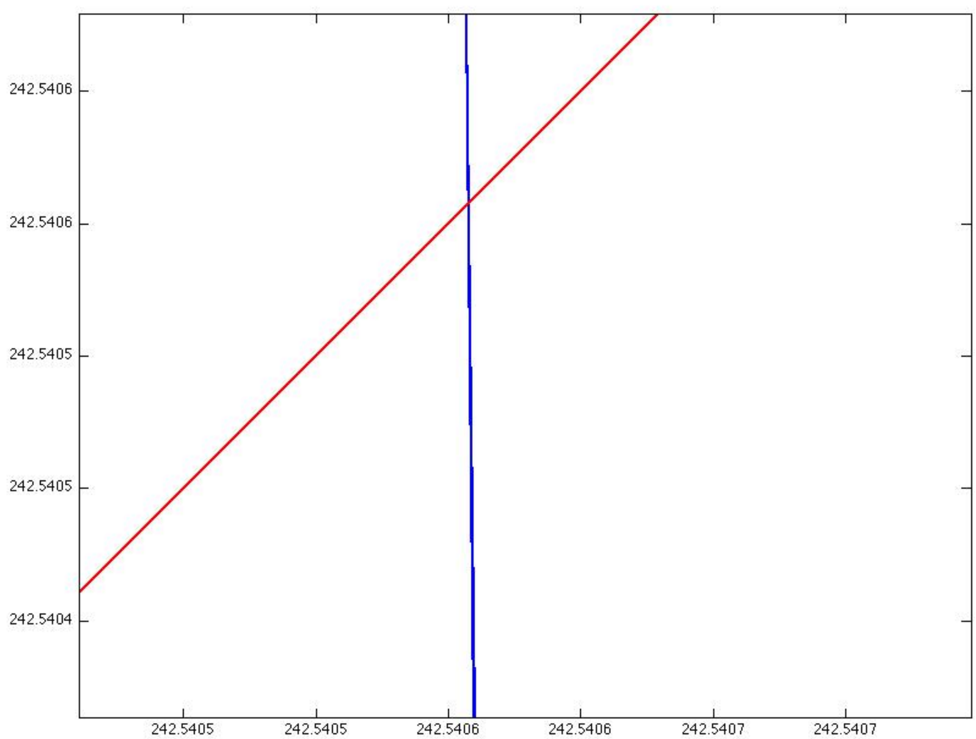


Figure 4.4: Aggregate Capital Stock at Equilibrium Using a Grid Search - zoomed about K^*

or the dynamic programming layer is not run long enough. In our method, we use 1000 iterations. In programs implementing algorithm 5 (or 4) the value function is iterated only 100 times. In partial justification, [4] continues to update the value function over time. That is, with each K_{next} value in 4.11, they update (r, w), government policy (τ, m), and the last value function becomes the initial value on this run with new calibration parameters. It is important to note that this adds error. That is, this convergence is only reproduced if we follow the same sequence of K_{next} values and rerun the program. And the program does not return $K_{converged} = 243.6$ in one run. If $K_{converged}$ is a value at equilibrium, it should be unchanged. In our case for each $K_{in} \in kGrid$, we reinitialize the value function with a zero vector and run [11] for 1000 iterations or until $V^{iteration+1} - V^{iteration}$ is small.[12]

Analysis of our results. For results in Figure 4.3, we present the following explanation. If K_{in} is too small then r (the interest rate) is too large which encourages agents to save and invest more so that the policy functions produce large household assets/wealth with density mean becoming large. Thus too little capital in the economy encourages more investment leading to more capital in the next iteration. Similarly, too much capital leads to less investment and smaller capital stocks in the next iteration. We think the intersection point with the 45 degree line in figure 4.4 needs to have slope less than 1 in absolute value for gradient algorithms like 4 and 5 to be stable. We can exploit the above relationship between r and K_{in} (and $K_{computed}$) to our advantage and use a faster algorithm inspired by the bisection or binary search method.

Algorithm 9. ***: A general algorithm to compute the Stationary Equilibrium using a binary search on aggregate capital stock grid***

Step 1: *Choose a wide interval for the aggregate capital stock* (K_{min}, K_{max}). *You can choose the same or slightly smaller interval than the household asset grid. We know from the above analysis that* $K_{computed} > K_{min}$ *and* $K_{computed} < K_{max}$.

Step 2: *Let* $K_{max_{new}} = (K_{min} + K_{max})/2$

Step 3: *Follow steps* $2 - 3f$ *in algorithm 8 to compute the stationary density and* $K_{computed}$ *value for the input,* $K_{max_{new}}$

Step 4: *if* $(K_{max_{new}} > K_{computed})$, *set* $K_{max} = K_{max_{new}}$; *else* $K_{min} = K_{max_{new}}$

Step 5: *if(*$\|K_{max} - K_{min}\| <$ *constant), stop.*

In step 1, we use an interval of $(50, 2900)$. This interval is comparable to the individual asset grid interval of $(0, 3000)$. In step 4, the statement $(K_{max_{new}} > K_{computed})$ implies that K^* is to the left of $K_{max_{new}}$, and K_{max} is updated. Likewise $(K_{max_{new}} > K_{computed})$, implies that K^* is to the right of $K_{max_{new}}$ and K_{min} is updated. In step 5, we choose the value of the constant to be 0.001. This choice is made to make the results comparable with K^* using the grid technique earlier. If no prior information is at hand, a more accurate choice can be made. The converged K^* value using this algorithm is in the interval $(242.5303, 242.5330)$.

[11] There is no justification for this number, but we have found that this is sufficient for the examples in this paper. The ideal number of runs is an open problem.

[12] We use convergence in norm

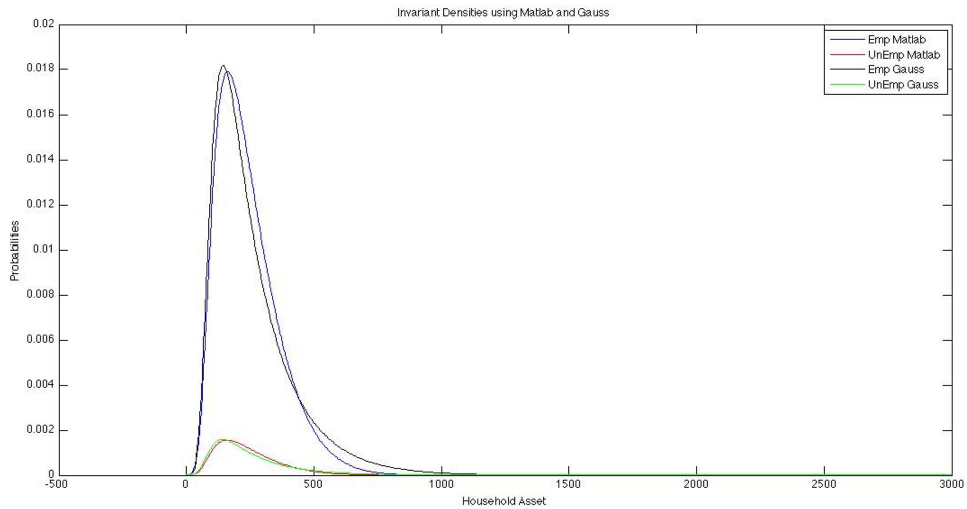


Figure 4.5: Density Comparison Using Algorithms 5 and 9

Figure 4.5 compares two plots - plot of the density our binary search method, and a plot using the code provided in GAUSS by [4]. For our method if we use $K^* = 242.5330$ (that upper limit value of the converged interval), we get $K_{computed} = 242.2606$ as the mean of the density. The absolute relative error is 0.1123%. In GAUSS using algorithm 5, we get an absolute relative error of 4.9% with $K^* = 243.7$ and $K_{computed} = 255.65$ as the mean of the density. Please also note that both densities can be thought of as piecewise or conditional on the employment status so that the sum of the areas under both curve adds to 1.

Table 4.2: Running Time Analysis

	Algorithm 9 (our method)	Algorithm 5
Running Time (hrs.)	2.35	3.49

The binary search algorithm 9 is run to produce the same precision as in algorithm 5 (i.e. within 0.1) to make a valid comparison. Both algorithms were run on a Intel (R) 2 Duo CPU T8300 2.40 GHz notebook. The initial interval in step 1 of algorithm 9 is relatively coarse $(200, 300)$ and is chosen to contain the aggregate capital stock of the representative[13] agent economy, 247.6. For every choice of $K_{max_{new}} \in (K_{min}, K_{max})$, we restart the value function with a zero vector and run until there is convergence. The running time of our binary search algorithm is smaller compared to algorithm 5. This is due to the enormous time saved by computing the stationary density using eigen-analysis and the Perron-Frobenius Theorem. Please note that the running time will increase if we choose a wider initial interval and aim for higher precision. Thus the running time of the wider initial interval, $(50, 2900)$,

[13]An economy with the same calibration but no shocks.

described earlier was 4.87 hours. The higher running time is attributed to the three decimal precision used as a stopping criteria in step 5 of algorithm 9. This wider initial interval does not require any prior information, especially - the aggregate capital stock value of the representative agent economy - as needed in algorithms described by [4] .

4.7 Time Iterative Technique

The algorithms discussed above are called fixed point techniques because they focus on finding the stationary point. We will now discuss the time iterative technique method. Here we want to compute the non-stationary state of an economy. We reconsider Example 4 with one important difference: the economy is not in stationary equilibrium. Recall that the households are allocated uniformly along an interval and are of measure one. For convenience of notation the time subscripts are dropped. The individual household now maximizes

$$V(\epsilon, a, \gamma) = \max_{c,a'} \left[u(c) + \beta E\{V(\epsilon', a', \gamma')\}|\epsilon\right], \tag{4.13}$$

$$\text{s.t.} \tag{4.14}$$

$$a' = \begin{cases} (1+(1-\tau)r)a + (1-\tau)w - c, & \text{if } \epsilon = e \\ (1+(1-\tau)r)a + m - c, & \text{if } \epsilon = u \end{cases} \tag{4.15}$$

$$a \geq a_{min} \tag{4.16}$$

$$\Pi(\epsilon'|\epsilon) = Pr\{\epsilon_{t+1} = \epsilon'|\epsilon_t = \epsilon\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix} \tag{4.17}$$

The distribution of $\gamma(\epsilon, a)$ is described by the following dynamics:

$$\gamma'(\epsilon', a') = \sum_{\epsilon=e,u} \pi(\epsilon', \epsilon)\gamma\left(\epsilon, a'^{-1}(\epsilon, a', \gamma)\right) \tag{4.18}$$

Note the increase in the individual state space by the infinite dimensional parameter γ. At each time instant, the households have to estimate γ'. They use this density parameter to predict the next period interest rates. These prices are equal to their respective marginal products:

$$r = \alpha\left(\frac{N}{K}\right)^{1-\alpha} - \delta \tag{4.19}$$

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}. \tag{4.20}$$

The aggregate consistency conditions hold:

$$\begin{aligned}
K &= \sum_{\epsilon\in\{e,u\}} \int_{a_{min}}^{\infty} a\ \gamma(\epsilon,a)\ \ da, \\
N &= \int_{a_{min}}^{\infty} \gamma(\epsilon=e,a)\ \ da, \\
C &= \sum_{\epsilon\in\{e,u\}} \int_{a_{min}}^{\infty} c(\epsilon,a)\ \gamma(\epsilon,a)\ \ da, \\
T &= \tau(wN+rK), \\
M &= \int_{0}^{\infty} m\gamma(\epsilon,a)\ \ da.
\end{aligned}$$

Note that the aggregate variables K (aggregate wealth), N (aggregate employment), C (aggregate consumption), T (total tax) and M (total unemployment benefits) are not constant and vary over time. There is no stationary equilibrium in this set up. However, a sequential competitive equilibrium exists such that:

1. Given prices $(r, w), (a', c')$ solves problem 4.13 for all $t \geq 0$.

2. Given prices (r, w), the firm maximizes profits so that 4.19 is satisfied for all $t \geq 0$.

3. For all $t \geq 0$, the labor market clears

$$N = \int_{a_{min}}^{\infty} \gamma(\epsilon = e, a)\ \ da,$$

and the goods market clears

$$C + K' = F(K, N) + (1 - \delta)K.$$

If we are able to find a fixed point such that $K = K'$, $N = N'$, etc, we would have stationary equilibrium. To solve this example using the current set up, we follow Krusell and Smith [7]. They argue that the households need γ' to approximate the next period interest rates. But the households only use the first moment, K', in approximating this value. In a way, the households only use partial information. They assume a law of motion H_K for the aggregate capital inspired by the derivation 3.15 in example 1.

$$\ln K' = \beta_0 + \beta_1 \ln K. \tag{4.21}$$

Algorithm 10. ***: Solving the economy given in example 4 using the Krusell and Smith method***

Step 1: Choose the initial distribution of assets γ_0 with mean K_0.

Step 2: Solve the consumer's optimization problem and compute $V(\epsilon, a, K)$.

Step 3: Simulate the dynamics of the distribution, Γ.

Step 4: *Use the time path for the distribution to estimate the law of motion for the moments* K.

Step 5: *Iterate until the parameters of* H_K *converge.*

Step 6: *Test the goodness of fit for* H_K. *If the fit is satisfactory, stop, otherwise increase* I *or choose a differential functional form for* H_K.

The functional form for 4.22 after the goodness of fit test was determined to be:

$$\ln K' = 0.041910 + 0.992388 \ln K. \tag{4.22}$$

This equation implies a stationary capital stock equal to $K^* = e^{\frac{0.041910}{1-0.992388}} = 246.1$ found by setting $K' = K$. The constant aggregate capital stock K^* along with a stationary density γ coming from the dynamics given in equation 4.18, and aggregate stationary employment N coming from the employment unemployment matrix given in 4.12 implies a stationary equilibrium. The value of $K^* = 246.1$ can be compared to the rest in table 4.6. Please note that based on figure 4.3, this value implies a lot of error.

Discussion. Krusell and Smith's time-iterative method [7] discussed above and the fixed point methods given by algorithms 4.6, 8 and 9 all give an approximate solution to example 4. Krusell and Smith's algorithm focuses on the non-stationary transition dynamics while we focus on the stationary equilibrium. The idea is to be at some equilibrium point, change a specific parameter (say tax rate), and then assess the path to reach the next equilibrium. [4] shows how to use transition dynamics to go from a non-stationary initial distribution to a stationary one. We are focusing on the parameters and analysis at stationary equilibrium only. We want to point out that computing stationary equilibrium from transition dynamics leads to errors - at least in the present case where we assume a functional form on aggregate capital (see equation 4.22). This is the reason why we emphasize the methods used in algorithms 8 and 9 which are accurate and robust. Of course there is a trade-off between accuracy and running time. Though our algorithms may not be appropriate for all economies, they inch us forward in the right direction.

CHAPTER 5

DYNAMICS OF A HETEROGENEOUS AGENT ECONOMY - MODEL WITH AGGREGATE AND IDIOSYNCRATIC UNCERTAINTY

As in the last chapter, three sectors exist in this economy: a continuum of households who maximize consumption and savings, a single firm - owned collectively by these households - that maximizes profits and is responsible for production, and a government who redistributes wealth by charging tax and paying unemployment benefits. In addition, this model includes aggregate (technological) and idiosyncratic (employment) uncertainty. Aggregate uncertainty with two realizations in relation to real economies could be interpreted as periods of growth and recession.

In literature, the aggregate shock z_t typically follows an $AR(1)$ process. If this is the case, this random process can be discretized in the following manner:

Markov Chain Approximation of $AR(1)$ Process as given in [4]:

Consider the process:

$$z_{t+1} = \rho z_t + \epsilon_t, \tag{5.1}$$

where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The unconditional mean and variance of the process are 0 and $\sigma_z^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$. [16] proposes to choose a uniform grid $\mathcal{Z} = [z_1, ..., z_m]$ whose upper endpoint is a multiple c (> 0) of the unconditional standard deviation, $z_m = c\sigma_z$, and the lower endpoint is $z_1 = -z_m$. For a given realization $z_i \in \mathcal{Z}$, the variable $z := \rho z_i + \epsilon$ is normally distributed with mean ρz_i and variance σ^2. Let dz be the midpoint between two consecutive grid points, then

$$Pr(z_j - dz \leq z \leq z_j + dz) = \Pi(z_j + dz) - \Pi(z_j - dz),$$

where $\Pi(\cdot)$ denotes the cumulative distribution function of the normal distribution with mean ρz_i and variance σ^2. Thus the post standardization probability to switch from the state z_i to z_j is $p_{ij} = \Phi(\frac{z_j - \rho z_i + dz}{\sigma_\epsilon}) - \Phi(\frac{z_j - \rho z_i - dz}{\sigma_\epsilon})$, where $\Phi(\cdot)$ denotes the standard normal distribution.

Let us assume that in the present model, the economy faces two aggregate shocks - good g and a bad b, and two idiosyncratic shocks - employment and unemployment. At the beginning of period t the aggregate (technology) shock z_t and the idiosyncratic shock ϵ_t are realized. The joint process of the two shocks, z_t and ϵ_t, can be written as a Markov process with a 4×4 transition matrix $\Pi(z_{t+1}, \epsilon_{t+1} | z_t, \epsilon_t)$. The transition matrix is defined as:

$$\Pi(z', \epsilon' | z, \epsilon) = Pr\{z_{t+1} = z', \epsilon_{t+1} = \epsilon' | z_t = z, \epsilon_t = \epsilon\} \tag{5.2}$$

$$= \begin{pmatrix} p_{z_{ge}z_{ge}} & p_{z_{ge}z_{gu}} & p_{z_{ge}z_{be}} & p_{z_{ge}z_{bu}} \\ p_{z_{gu}z_{ge}} & p_{z_{gu}z_{gu}} & p_{z_{gu}z_{be}} & p_{z_{gu}z_{bu}} \\ p_{z_{be}z_{ge}} & p_{z_{be}z_{gu}} & p_{z_{be}z_{be}} & p_{z_{be}z_{bu}} \\ p_{z_{bu}z_{ge}} & p_{z_{bu}z_{gu}} & p_{z_{bu}z_{be}} & p_{z_{bu}z_{bu}} \end{pmatrix} \tag{5.3}$$

The entry, $p_{z_{gu}z_{ge}}$, for example, denotes the probability of transiting from state z_{gu} (unemployment state in a good economy) to the state z_{ge} (employment state in a good economy). We assume the households know the law of motion of both ϵ_t and z_t. With aggregate shock, the firm's production levels are influenced - there is higher production in good times and vice versa. Likewise the employment status of the current period affects household savings and consumption. As usual, the firm is interested in maximizing profits and the households are focused on maximizing lifetime utility. And since the firm is owned by households collectively, its problem remains static. As in previous models, we focus on solving the household problem.

5.1 The Household's Problem

There is a continuum of households distributed on an interval with measure one. The individual household maximizes its utility:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]. \tag{5.4}$$

We assume that this optimization problem can be represented[1] as a dynamic programming problem. [**PMB:** The mbox steps on the equation number in eq 5.5. You will have to use "multline" to get an equation line break.]

$$V(\epsilon_t, a_t, z_t, \gamma_t) = \max_{c,a'} \left[u(c_t) + \beta E\{V(\epsilon_{t+1}, a_{t+1}, z_{t+1}, \gamma_{t+1}\} | \epsilon_t, z_t, \gamma_t\right], \quad \text{with constraints} \tag{5.5}$$

$$a_{t+1} = \begin{cases} (1 + (1 - \tau_t) r_t) a_t + (1 - \tau_t) w_t - c_t, & \text{if } \epsilon_t = e \\ (1 + (1 - \tau_t) r_t) a_t + m_t - c_t, & \text{if } \epsilon_t = u \end{cases} \tag{5.6}$$

$$a_t \geq a_{min} \tag{5.7}$$

$$u(c_t) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0 \tag{5.8}$$

where τ_t, m_t, r_t, w_t, a_t, c_t are government tax rate, unemployment policy, interest and wage rate, individual wealth and consumption, respectively for period t. We note the increase in the state space of the value function V in equation 5.5 above in comparison to the value function equation 4.10 of the previous model. Two parameters are added - the aggregate shock argument z_t and the wealth distribution γ_t. An individual household is unable to

[1] Not all variants of the lifetime utility optimization problem may be represented as a dynamic programming problem. For the problems that can be represented, existence and uniqueness of the value function has to be justified. The present paper consists of elementary models from [4] which cites [10] for this justification.

infer the value of the next-period aggregate capital stock K_{t+1} to predict r_{t+1} from its own decision. It needs to know how all the other (infinitely-numbered) households in the economy decide and how they save via γ_t in order to compute K_{t+1} and hence, r_{t+1}. Since γ_t is a continuous function, approximating this infinite dimensional parameter in every iteration makes this problem not just difficult but daunting!

5.2 The Firm's Problem

There is a single firm owned collectively by households. Production is expressed in analytical form using the Cobbs-Douglas production function:

$$F(K_t, N_t) = Dz_t K_t^{\alpha} N_t^{1-\alpha} \ , \ \alpha \in (0,1), \tag{5.9}$$

where D is constant, z_t, K_t, and N_t are the aggregate shock, capital and labor, respectively. Define output as:

$$Y_t \equiv F(K_t, N_t) + (1-\delta)K_t$$

Given the interest rate, r_t and the wage rate, w_t, the firm has to decide on the optimal amount of aggregate capital, K_{t+1} and labor, N_t to maximize profit:
Firm's problem:

$$\max_{(K_{t+1}, N_t)} Y_t - r_t K_{t+1} - w_t N_t, \tag{5.10}$$

The interest and wage rates are set by taking partial derivatives:

$$\begin{aligned} r_t &= \frac{\partial Y_t}{\partial K_t} && (5.11)\\ &= \alpha z_t \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta && (5.12)\\ w_t &= \frac{\partial Y_t}{\partial N_t} && (5.13)\\ &= (1-\alpha) z_t \left(\frac{N_t}{K_t}\right)^{\alpha}, && (5.14) \end{aligned}$$

where $\delta \in [0,1]$ is the rate at which capital depreciates.

5.3 The Government's Problem

The role of the government is to redistribute capital by charging an income tax and giving out unemployment compensation. The government has to balance its budget using the condition:

$$T_t = M_t.$$

The government policy is characterized by a constant replacement ratio $\zeta = m_t/(1-\tau_t)w_t$. So m_t has to be adjusted based on the wage rate w_t.
Figure 5.1 summarizes each sector's decision process using a model diagram. The demand d and supply s superscripts on aggregate capital stock and labor are not mentioned in algorithms or model definitions as we only use the converged variables (i.e. $K_t^d = K_t^s \equiv K_t$, etc).

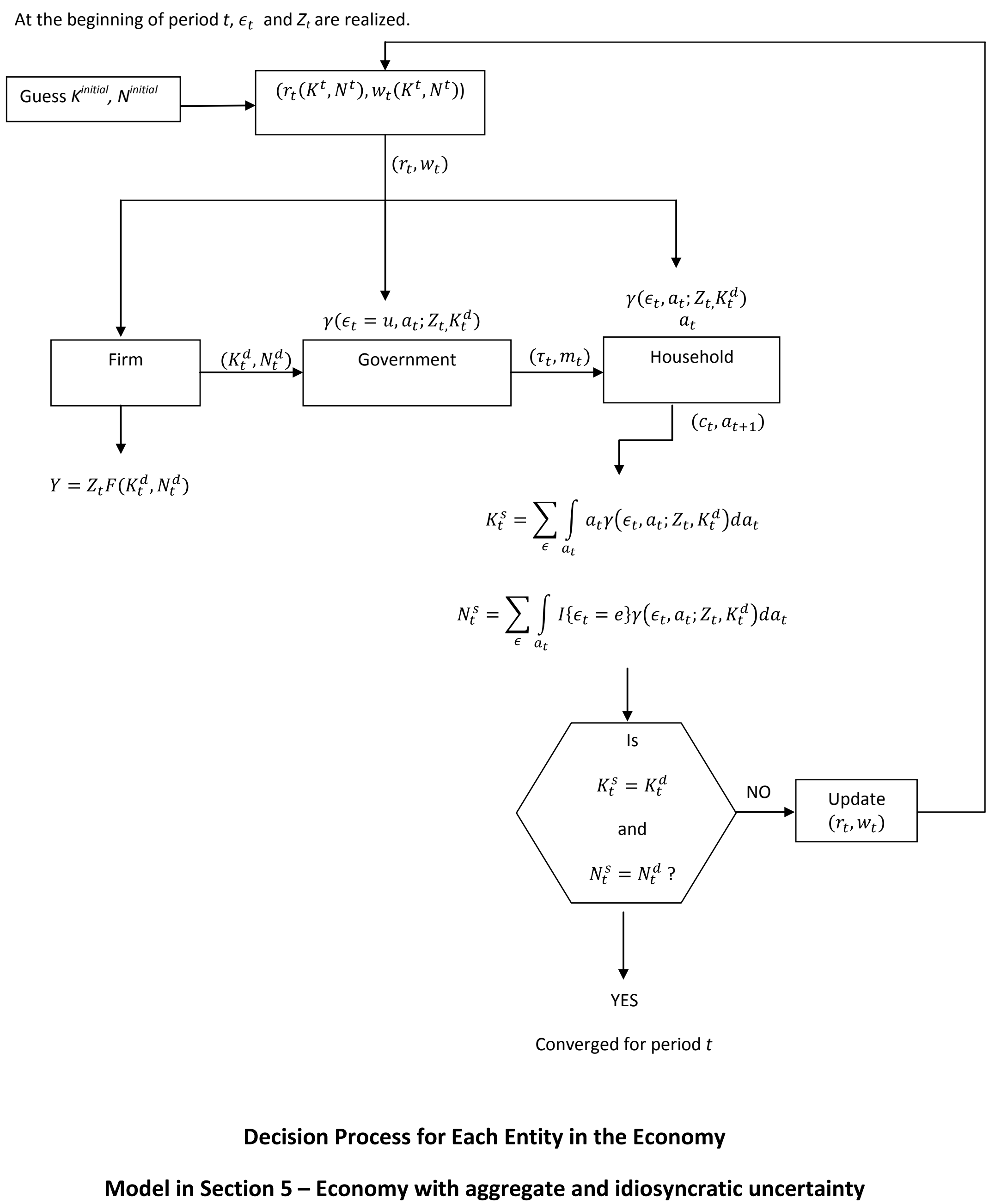


Figure 5.1: Decision process and arguments of each entity in a economy with aggregate and idionsyncratic uncertainty.

5.4 The Competitive Equilibrium

There is no stationary equilibrium in the current setup. A sequential competitive equilibrium consists of admissible allocation $(\epsilon_t, a_t)_{t\geq 0}$ for all households and price processes $(w_t, r_t)_{t\geq 0}$ such that:

1. Given prices $(w_t, r_t)_{t\geq 0}$, $(a_{t+1}, c_t)_{t\geq 0}$ solves the household optimization problem 5.4.
2. Given prices $(w_t, r_t)_{t\geq 0}$, the firm maximizes its profits given in equation 5.10.
3. The labor market clears

$$N_t = \int_a \gamma_t(\epsilon_t = e, a_t; z_t, K_t) da,$$

and the goods market clears, $\forall t \geq 0$

$$C_t + K_{t+1} = z_t F(K_t, N_t) + (1-\delta)K_t.$$

An aggregate distribution or density (considered here) is defined over the individual states across the households. An individual state space consists of sets $(\epsilon, a, z) \in \chi = \{e, u\} \times [a_{min}, \infty) \times \{g, b\}$. The distribution of the individual states (ϵ_t, a_t) for given aggregate state variables (z_t, K_t) in period t is denoted by $\gamma_t(\epsilon_t, a_t; z_t, K_t)$. The agent has to determine K_{t+1} through the non-stationary aggregate density, $\gamma_t(\epsilon_t, a_t; z_t, K_t)$ to approximate r_{t+1} and w_{t+1}. The dynamics of the distribution of the individual states are described by the following equations:

$$\gamma_{t+1}(\epsilon_{t+1}, a_{t+1}; z_{t+1}, K_{t+1}) = \sum_{\epsilon} \Pi(z_{t+1}, \epsilon_{t+1} | z_t, \epsilon_t) \gamma_t(\epsilon_t, a_t; z_t, K_t), \tag{5.15}$$

where $a_{t+1} = a_{t+1}(\epsilon_t, a_t; z_t, K_t)$, and

$$K_{t+1} = \sum_{\epsilon} \int_a a_{t+1}\ \gamma_t(\epsilon_t, a_t; z_t, K_t)\ \ da.$$

The shocks (z, ϵ) follow a Markov structure, $\Pi(z', \epsilon' | z, \epsilon)$, with transition matrix given in 5.2.

The aggregate variables are written as an expectation with respect to the aggregate distribution:

$$\begin{aligned}
K_t &= \sum_{\epsilon} \int_a a_t\ \gamma_t(\epsilon_t, a_t; z_t, K_t)\ \ da, \\
N_t &= \int_a \gamma_t(\epsilon_t = e, a_t; z_t, K_t)\ \ da, \\
C_t &= \sum_{\epsilon} \int_a c_t(\epsilon_t, a_t; z_t, K_t) \gamma_t(\epsilon_t, a_t; z_t, K_t)\ \ da, \\
T_t &= \tau_t(w_t N_t + r_t K_t), \\
M_t &= \int_a m_t \gamma_t(\epsilon_t = u, a_t; z_t, K_t)\ \ da.
\end{aligned}$$

5.5 Discussion

Due to the presence of aggregate uncertainty and idiosyncratic uncertainty, there are three major changes in this model:

1. We have fluctuating employment levels.

2. The goal of the households in every period is to predict factor prices, i.e. next period interest rate, r_{t+1}. To do this, they need to estimate, K_{t+1} from γ_t. What if the agents use only partial information such as the first moment (aggregate capital) instead of the entire density? That is, what if the agents assume the law of motion for the state variable, K_t inspired by 6, particularly:

$$\ln K_{t+1} = \begin{cases} \xi_{0g} + \xi_{1g} \ln K_t & \text{if } z = z_g \\ \xi_{0b} + \xi_{1b} \ln K_t & \text{if } z = z_b \end{cases},$$

   Then the goal of the agent becomes to estimate parameters: ξ_{0g}, ξ_{1g}, ξ_{0b}, and ξ_{1b}. This type of approach is standard [4], [7] and others have used it as well. The goal is to assume an analytic form on the evolution of the aggregate capital stock, K_t. The parameters of this analytic equation are updated and estimated in every iteration. A positive goodness of fit test terminates this process. If the fit is mediocre, then an alternate analytic form involving higher moments of the wealth distribution is used. In general, agents can characterize the wealth distribution function: $\Gamma_t(\cdot)$ by say I statistics $q = (q_1, ..., q_I)$. In this case, the value function is restated as

$$V(\epsilon_t, a_t, z_t, q_t) = \max_{c_t, a_{t+1}} \left[\frac{c_t^{1-\eta}}{1-\eta} + \beta E\{V(\epsilon_{t+1}, a_{t+1}, z_{t+1}, q_{t+1} | \epsilon_t, z_t, q_t\} \right]$$

   where

$$q = H_I(q, z). \tag{5.16}$$

The following algorithm computes the dynamics in the heterogenous-agent economy with aggregate and idiosyncratic uncertainty using the approach given in [7].

Algorithm 11 (Computation of Competitive Equilibrium)**.**

1. *Compute aggregate next period employment N as a function of current productivity $z : N = N(z)$.*

2. *Choose the order I of moments m.*

3. *Guess a parameterized functional form for H_I in Equation 5.16 and choose initial parameters of H_I.*

4. *Solve the consumer's optimization problem and compute $V(\epsilon, a, z, m)$.*

5. *Simulate the dynamics of the distribution function.*

6. *Use the time path for the distribution to estimate the law of motion for the moments m.*

7. *Iterate until the parameters of H_I converge.*

8. *Test the goodness of fit for H_I using, for example, $R-Square$. If the fit is satisfactory, stop, otherwise increase I or choose a different functional form for H_I.*

The issue with this last model is that the solution is difficult to assess if a minor variation is made to the model. In addition, the computation time is typically a matter of weeks not days. Solutions that involve setting a fine grid point on aggregate capital stock are more accurate but also come at a high computational cost. [12] introduces a faster technique using projections. The computer solution is nonlinear in idiosyncratic shocks but linear in aggregate shocks. [13] uses a parameterization of the cross sectional distributions which enable the use of quadrature instead of Monte Carlo techniques to improve on accuracy. We propose using the algorithms presented in the last chapter - grid and binary search based algorithms. They avoid the functional form assumption on the aggregate capital. Furthermore, we solve for all possible outcomes of the aggregate and idiosyncratic shocks. This eliminates the wealth density argument in the value function for the households. Our idea is to set up Markov matrices for all possible random states and find stationary densities at various initial conditions of the aggregate capital stock. The wealth density is computed more quickly using eigen analysis and the Perron-Frobenius theorem. We set a grid on the aggregate capital stock and look for the value in this grid that returns itself (with some acceptable error) at equilibrium as the first moment of the wealth density.

5.6 Our method

Algorithm 12. ***: A general algorithm to compute the Stationary Equilibrium using a grid on aggregate capital stock***

Step 1: *Set a grid on the aggregate capital stock, KGrid.*

Step 2: *For each aggregate shock, z, for each value, $K_{in} \in KGrid$, do the following sub-steps:*

Step 2a: *Compute the employment, $N(z)$.*

Step 2b: *Determine the factor prices (r, w) using marginals, and the government policy as a function of the aggregate shock and employment.*

Step 2c: *For each employment state, ϵ*

Step 2d: *Compute the household's decision rules, $a' = a'(a, \epsilon; z)$ and $c(a, \epsilon; z)$, using the computed value function $V(a, \epsilon, z)$*

Step 2e: *Compute the invariant density of assets for the employed and unemployed agents using the Perron-Frobenius Theorem.*

Step 3: *Compute the mean of the above density for all aggregate shock states. Call this aggregate capital stock $K_{computed}$.*

Step 4: *Stop if $|K_{in} - K_{computed}| < \epsilon$*

In step 2e, we set up the transition matrices by constructing a $4 * aGrid \times 4 * aGrid$ $a - a'$ individual policy transition matrix A from the optimal policy derived using the value function iteration. The transition matrix $\in \mathbb{R}^{4m} \times \mathbb{R}^{4m}$ for (a, ϵ, z) is constructed as follows:

$$\begin{aligned} P(a_{t+1}, \epsilon_{t+1}, z_{t+1}|(a_t, \epsilon_t, z_t)) &= P(a_{t+1}|\epsilon_{t+1}, z_{t+1}, (a_t, \epsilon_t, z_t))P(\epsilon_{t+1}, z_{t+1}|(a_t, \epsilon_t, z_t)) \\ & \quad a_{t+1} \text{ is independent of } \epsilon_{t+1} \text{ and } z_{t+1} \\ &= P(a_{t+1}|(a_t, \epsilon_t, z_t))P(\epsilon_{t+1}, z_{t+1}|(a_t, \epsilon_t, z_t)) \end{aligned}$$

If we use our results from the last chapter: i.e. If K_{in} is too small then r (the interest rate) is too large which encourages agents to save and invest more. The policy functions produce large household assets/wealth with a large density mean. Thus too little capital in the economy encourages more investment leading to more capital in the next iteration. Similarly, too much capital leads to less investment and smaller capital stocks in the next iteration. Using this reasoning, we can avoid the grid approach mentioned above and reuse the binary search algorithm from the last chapter.

Algorithm 13. *: **A general algorithm to compute the Stationary Equilibrium using a binary search on aggregate capital stock grid***

Step 1: *Choose a wide interval for the aggregate capital stock (K_{min}, K_{max}). You can choose the same or slightly smaller interval than the household asset grid. We know from the above analysis that $K_{computed} > K_{min}$ and $K_{computed} < K_{max}$.*

Step 2: *Let $K_{max_{new}} = (K_{min} + K_{max})/2$*

Step 3: *Follow steps $2 - 3f$ in algorithm 12 to compute the stationary density and $K_{computed}$ value for the input, $K_{max_{new}}$*

Step 4: *if $(K_{max_{new}} > K_{computed})$, set $K_{max} = K_{max_{new}}$; else $K_{min} = K_{max_{new}}$*

Step 5: *if $(\|K_{max} - K_{min}\| < constant)$, stop.*

5.6.1 Stationary Equilibrium using our method

Since we solve for all possible outcomes of the aggregate and idiosyncratic shocks instead of simulating the economy over time as done in section 5.5, there exists a stationary equilibrium. The conditions for this state are defined below.

a) For each aggregate shock z, a given set of factor prices (r, w), government policy (m, τ), a value function $V(a, \epsilon, z)$, a sequence of individual decision rules $a' = a'(a, \epsilon, z)$ and $c(a, \epsilon, z)$ that solve the household optimization problem 5.4, and a stationary probability distribution function $\Gamma(a, \epsilon, z)$ or equivalently an invariant density function $\gamma(a, \epsilon, z)$ for household wealth exist.

b) The distribution of (a, ϵ, z) is stationary. The aggregate capital K, aggregate consumption C, and aggregate employment N are constant.

c) Factor prices are equal to their respective marginal products:

$$\begin{aligned} r &= \alpha\left(z\frac{N}{K}\right)^{1-\alpha} - \delta \\ w &= (1-\alpha)z\left(\frac{K}{N}\right)^{\alpha}. \end{aligned}$$

d) The government's budget balances: $M = T$.

e) The aggregate consistency conditions hold:

$$\begin{aligned} K &= \sum_{z\in\{z_g,z_b\}}\sum_{\epsilon\in\{e,u\}}\int_{a_{min}}^{\infty} a\,\gamma(a,\epsilon,z)\ \ da, \\ N &= \sum_{z\in\{z_g,z_b\}}\int_{a_{min}}^{\infty} \gamma(a,\epsilon=e,z)\ \ da, \\ C &= \sum_{z\in\{z_g,z_b\}}\sum_{\epsilon\in\{e,u\}}\int_{a_{min}}^{\infty} c(a,\epsilon,z)\,\gamma(a,\epsilon,z)\ \ da, \\ T &= \tau(wN + rK), \\ M &= \sum_{\epsilon\in\{z_g,z_b\}}\sum_{\epsilon\in\{e,u\}}\int_{a_{min}}^{\infty} m\,\gamma(a,\epsilon=u,z)\ \ da, \end{aligned}$$

and finally,

f) The goods market clears:

$$C + K' = F(K,N) + (1-\delta)K,$$

where

$$K' = \sum_{z\in\{z_g,z_b\}}\sum_{\epsilon\in\{e,u\}}\int_{a_{min}}^{\infty} a'\,\gamma(a,\epsilon,z)\ \ da$$

and

$$F(K_t,N_t) = zK^{\alpha}N^{1-\alpha}\ ,\ \alpha\in(0,1).$$

Recall that K and N are the aggregate capital and employment, respectively, C is the aggregate consumption for the household, and T and M are the aggregate tax received and unemployment benefits paid out by the government, respectively.

Before we implement our algorithm for a new example, we want to test it. If we use an appropriate calibration in this model, we could theoretically reproduce the results - stationary wealth density - from example 4. Here are the initial conditions

Example 5. *Let the agent utility function be given by*

$$u(c_t) = \frac{c_t^{1-\eta}}{1-\eta},\ \ \eta > 0$$

with $\eta = 2.0$

$$\Pi(\epsilon'|\epsilon) = \begin{pmatrix} 0.500 & 0.500 \\ 0.0435 & 0.9565 \end{pmatrix}. \tag{5.17}$$

The firm's production function as described in 4.5 with $\alpha = 0.36$, *i.e.*

$$F(K_t, N_t) = K_t^{0.36} N_t^{1-0.36}.$$

Let the individual discounting factor, β, *and capital depreciation rate,* δ, *be* 0.995 *and* 1, *respectively. Let the aggregate shock value be fixed at* 1 *i.e.* $(z_g = z_b = 1)$. *We set the stochastic transition matrix for* (a, ϵ, z) *using the one above, 5.17 :*

$$\Pi(z', \epsilon'|z, \epsilon) = Pr\{z_{t+1} = z', \epsilon_{t+1} = \epsilon'|z_t = z, \epsilon_t = \epsilon\} \tag{5.18}$$

$$= \begin{pmatrix} 0.9565 & 0.0435 & 1.0000 & 0 \\ 0.5000 & 0.5000 & 0 & 1.0000 \\ 1.0000 & 0 & 0.9565 & 0.0435 \\ 0 & 1.0000 & 0.5000 & 0.5000 \end{pmatrix} \tag{5.19}$$

There was a minor adjustment made while reproducing results. The perturbation added to satisfy conditions in the Perron-Frobenius Theorem had to be adjusted to 10^{-10} instead of the minimum of the wealth transition matrix. The choice of this constant is made such that it is smaller than the minimum of the individual wealth policy matrices in both algorithms - from this chapter and the last chapter. The solution - the stationary density at stationary equilibrium - to this example is given in Figure 5.2. Both Figures 5.2 and 5.3 have similar plots. The means - the aggregate capital stock - using an economy with just employment uncertainty, and an economy with technological and employment uncertainty were 242.5 and 242.3, respectively. Both means were computed using their respective stationary densities. Since the means are relatively close, we proceed to the next example.

Example 6. *Let the agent utility function be given by*

$$u(c_t) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0$$

with $\eta = 1.5$.

Let the firm's production function be described by

$$F(K_t, N_t) = z_t K_t^{\alpha} N_t^{1-\alpha},$$

where $\alpha = 0.36$. *Let the individual discounting factor,* β, *and capital depreciation rate,* δ, *be* 0.96 *and* 0.1, *respectively.*

Let the technology level be set to $z_g = 1.03$ *in good times and* $z_b = 0.97$ *in bad times. The calibration is such that the average duration of a boom or recession is* 5 *years. The transition matrix for technology is set to*

$$F_z(z'|z) = \begin{pmatrix} 0.80 & 0.20 \\ 0.20 & 0.80 \end{pmatrix}. \tag{5.20}$$

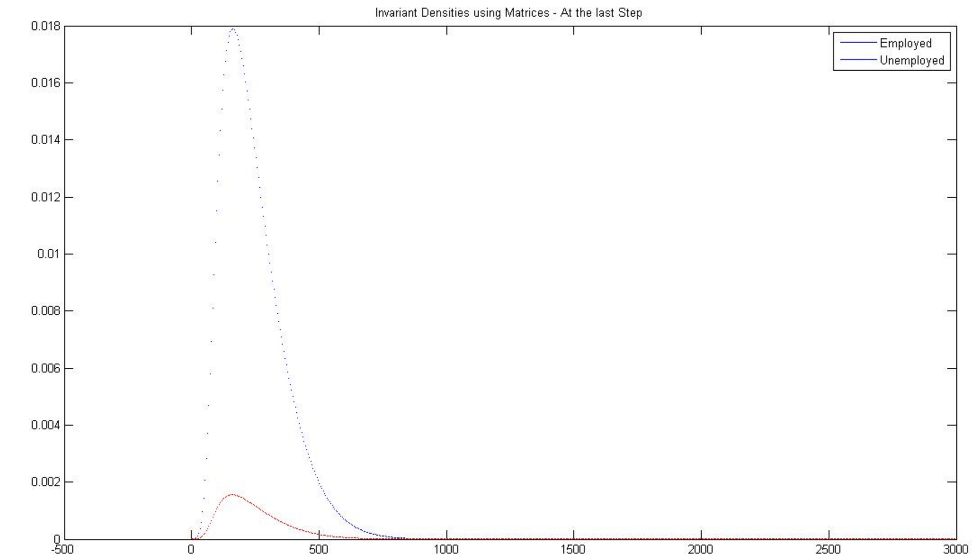


Figure 5.2: Density Comparison - using calibration from the previous chapter - Economy with both - Tech. and Emp. - Shocks

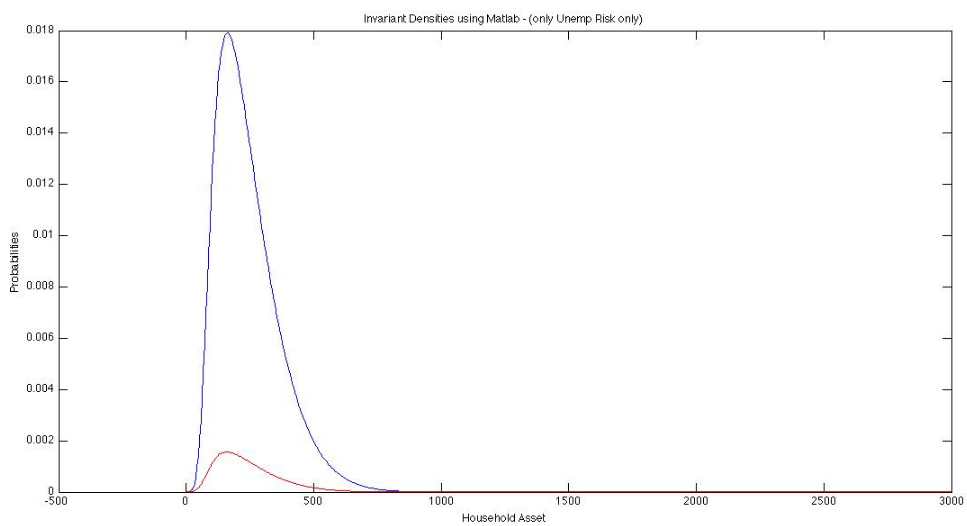


Figure 5.3: Density Comparison - Reproduced Plot from the previous chapter - Economy with Emp. Shock only

The eigenvector of 5.20 is such that equal amount of time is spent in each of the two aggregate states. The following conditional employment probabilities are used from [17] considering the annual employment mobility for the US economy:

$$F_{gg}(\epsilon'|z' = z_g, z = z_g, \epsilon) = \begin{pmatrix} 0.9615 & 0.0385 \\ 0.9581 & 0.04919 \end{pmatrix},$$

$$F_{bb}(\epsilon'|z' = z_b, z = z_b, \epsilon) = \begin{pmatrix} 0.9525 & 0.0475 \\ 0.3952 & 0.6048 \end{pmatrix}.$$

These employment probabilities imply ergodic distributions with unemployment rates $u_g = 3.86\%$ *and* $u_b = 10.73\%$ *in good and bad times, respectively. The conditional employment probabilities for the transition from good to bad times, are calibrated such that all unemployed agents stay unemployed and that the unemployment rate is* u_b *in the next period. The following constraint is used:*

$$u_z \frac{P_{z_u z'_u}}{P_{zz'}} + (1 - u_z) \frac{P_{z_e z'_u}}{P_{zz'}} = u_{z'}.$$

Similarly using the above constraint, the conditional employment probabilities for the transition from bad to good times are calibrated. In this case the probabilities are set such that all unemployed agents stay unemployed and that the unemployment rate is u_g *in the next period. Using these equations we set the following matrices.*

$$F_{gb}(\epsilon'|z' = z_b, z = z_g, \epsilon) = \begin{pmatrix} (1 - u_b)/(1 - u_g) & 1 - (1 - u_b)/(1 - u_g) \\ 0 & 1 \end{pmatrix},$$

$$F_{bg}(\epsilon'|z' = z_b, z = z_g, \epsilon) = \begin{pmatrix} 1 & 0 \\ (1 - ug/ub) & ug/ub \end{pmatrix}.$$

Thus, the 4×4 *transition matrix* $\Pi(z_{t+1}, \epsilon_{t+1}|z_t, \epsilon_t)$ *is defined as:*

$$\Pi(z', \epsilon'|z, \epsilon) = Pr\{z_{t+1} = z', \epsilon_{t+1} = \epsilon'|z_t = z, \epsilon_t = \epsilon\} \tag{5.21}$$

$$= \begin{pmatrix} 0.80 * F_{gg} & 0.20 * F_{bg} \\ 0.20 * F_{bg} & 0.80 * F_{bb} \end{pmatrix} \tag{5.22}$$

In our case, we obtain:

$$\Pi(z', \epsilon'|z, \epsilon) = Pr\{z_{t+1} = z', \epsilon_{t+1} = \epsilon'|z_t = z, \epsilon_t = \epsilon\} \tag{5.23}$$

$$= \begin{pmatrix} 0.7692 & 0.0308 & 0.1857 & 0.0143 \\ 0.7665 & 0.0335 & 0 & 0.2000 \\ 0.2000 & 0 & 0.7620 & 0.0380 \\ 0.1280 & 0.0720 & 0.3162 & 0.4838 \end{pmatrix} \tag{5.24}$$

The following set of equations describes the household maximization problem and the constraint equations:

The households are uniformly distributed with measure one. Given the factor prices, (r, w), and government policy (τ, b), the household maximizes

$$V(a, \epsilon, z) = \max_{c,a'} \left[u(c) + \beta E\{V(a', \epsilon', z'\}|\epsilon, z\right], \tag{5.25}$$

$$s.t. \tag{5.26}$$

$$a' = \begin{cases} (1 + (1 - \tau)r)a + (1 - \tau)w - c, & \text{if } \epsilon = e \\ (1 + (1 - \tau)r)a + m - c, & \text{if } \epsilon = u \end{cases} \tag{5.27}$$

$$a \geq a_{min}, \tag{5.28}$$

along with the stochastic transition matrix, 5.24.

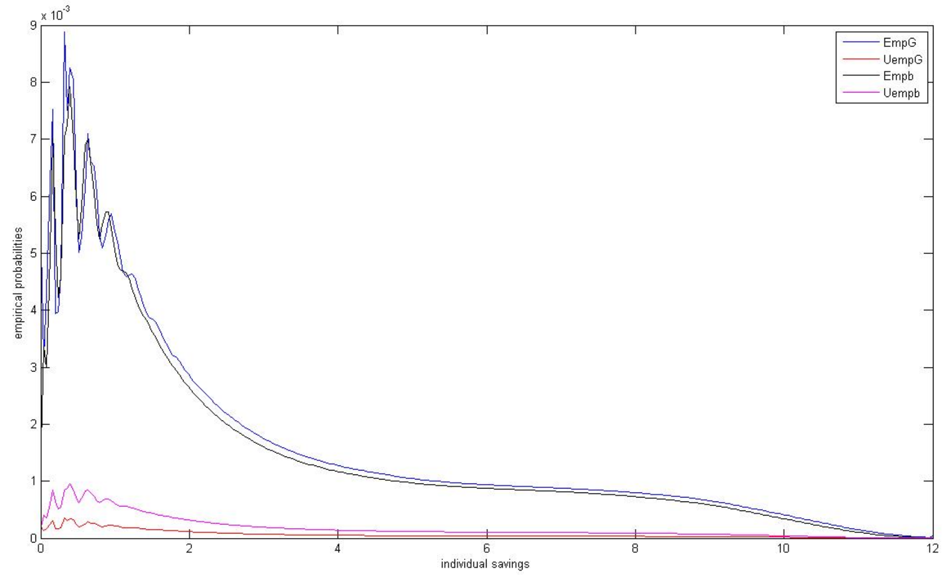


Figure 5.4: Individual Wealth Density Plots Economy with both - Tech. and Emp. - Shocks

The plot of the stationary density is given in figure 5.4. The aggregate capital stock value passed in using algorithm 13 was $K_{max_{new}} = 2.9521$*, and the computed value was* $K_{computed} = 2.9515$ *at convergence. Note that according to the densities in good and bad times, there is a higher proportion of unemployed households with lower asset wealth in a recession compared to a boom. Likewise larger proportions of employed households choose to save more in bad times. At stationary equilibrium the densities in the good and bad economies can be interpreted as follows: the distribution of assets is constant for both employed and unemployed agents given the state of the economy, and the number of employed and unemployed agents are also constant. Furthermore the first derivatives of the wealth densities are not smooth in the interval of* $(1, 2.5)$*. This is a minor fix - we can increase the number of grid points on the individual asset space or use interpolation or projection on the stationary wealth density. [4] solves this example by using the time-iterative technique* *focusing on transition dynamics and reports similar erratic behavior in the density in the interval* $(1, 2.5)$.

CHAPTER 6

FUTURE WORK

There are a number of ways to proceed. In the immediate future, the best way to test the robustness of our algorithm is to apply it to variants of the proposed economies. A wider individual wealth grid size is a characteristic of a richer economy. This increases the size of the individual wealth policy matrix, and consequently, the cross-sectional densities and their moment calculations become cumbersome. Several authors have used approximating functions in projection method, we propose using a numerical procedure such as principal component analysis (PCA) to compute the moments of the intermediary stationary distributions. This is a suitable solution to our method, especially in reducing running time. That is, we would like to project the high-dimension individual policy matrix to a smaller space and compute moments with 'acceptable' error.

A brief description of PCA: Let $X \equiv Pr\,(a', \epsilon' | a, \epsilon)$ the transition matrix corresponding to the stationary density. $X \in \mathbb{R}^{2m \times 2m}$, where m is the size of the finer asset grid. Let $z \in \mathbb{R}^{2m \times d}$ and $B \in \mathbb{R}^{2m \times d}$, where $d << m$ and $z = XB$. Principal component analysis, rotates the coordinate axes in order to have the new uncorrelated coordinates, principal components, with certain optimal variance properties. So we could do analysis on a lower dimension on z instead of doing analysis on X. Columns of z are uncorrelated and capture most variation of data in X. Consequently statistics can computed at the lower dimension with less percent error. A choice of d along with the trade-off between error and computational time should be made. PCA helps us choose the matrix B.

An alternative to dimension reduction (PCA) is to exploit the sparsity of the matrix directly while computing the stationary probabilities. An educated decision would have to made to weigh running time versus precision of each technique.

BIBLIOGRAPHY

[1] Aiyagari, R.S. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, Vol. 109, pp. 659-684.

[2] Aiyagari, R.S. (1995), "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints and Constant Discounting," *Journal of Political Economy*, Vol. 103, pp. 1158-1175.

[3] Den Haan, W.J. (1997) "Solving Dynamic Models with Aggregate Shocks and Heterogenous Agents," *Macroeconomic Dynamics*, Vol. 1, pp. 335-386.

[4] Heer, B. and Maubner, A. (2005), *Dynamic General Equilibrium Modeling: Computational Methods and Applications*, Springer, Berlin.

[5] Huggett, M. (1993) "The risk-free rate in heterogenous-agent incomplete-insurance economics," *Journal of Economic Dynamics and Control*, Vol. 17, pp. 953-969.

[6] İmrohoroğlu, A. et al. (1995) "A life cycle analysis of social security," *Economic Theory*, Vol. 6, pp. 83-114.

[7] Krusell, P. and Smith, A. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", *Journal of Political Economy*, Vol. 106, pp. 867-896.

[8] Ríos-Rull, J. V., (1997), "Computation of equilibria in heterogenous agent models". *Federal Bank of Minneapolis Staff Report*, Vol. 231, pp. 238-264.

[9] Ríos-Rull, J. V., (1999), "Computation of Equilibria in Heterogenous-Agent Models" in Mormon, R. and A. Scott (eds.), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, Oxford.

[10] Stokey, N.L and Lucas, Jr., R.E. (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, Massachusetts.

[11] Shorack, G.R., 2000. *Probability for Statisticians*, Springer-Verlag, New York, NY.

[12] Reiter, M (2009), "Solving Heterogeneous-Agent Models by Projection and Perturbation," *Journal of Economic Dynamics and Control*, Vol. 33(3), pp. 649-665.

[13] Den Haan, W.J., Algan, Y., and Allais, O., 2008. "Solving heterogenous-agent models with parameterized cross-section distributions," *Journal of Economic Dynamics and Control*, Vol. 32(3), pp. 875-908.

[14] Den Haan, W.J., Rendahl, P., 2009. "Solving the incomplete markets model with aggregate uncertainty using explicit aggregation". *Journal of Economic Dynamics and Control*, doi:10.1016/j.jedc.2008.12.008.

[15] Miao, Jianjun, 2005. "Competitive equilibria of economies with a continuum of consumers and aggregate shocks," *Journal of Economic Theory*, Vol. 128, pp. 274-298.

[16] Tauchen, George, 1986. "Finite State Markov-Chain Approximations to Univariate and Vector Autoregression", *Economics Letters*, Vol. 20, pp. 177-181.

[17] Castañeda, Ana, Javier Díaz-Giménez and Jóse-Victor Ríos-Rull, 1998b. "Exploring the income distribution business cycle dynamics", *Journal of Monetary Economics*, Vol. 42, pp. 93-130.

[18] Maliar, L., Maliar S., Valli, F., 2009. "Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm", *Journal of Economic Dynamics and Control*, doi:10.1016/j.jedc.2009.03.009.

[19] Young, E.R., 2009. "Solving the incomplete market model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations", *Journal of Economic Dynamics and Control*, doi:10.1016/j.jedc.2008.11.010.

[20] Algan, Y., Allais, O., den Haan, W.J., 2009. "Solving the incomplete markets model with aggregate uncertainty using parameterized cross-sectional distributions". *Journal of Economic Dynamics and Control*, doi:10.1016/j.jedc.2009.03.010.

[21] Preston, B., Roca, M., 2006. "Incomplete markets, heterogeneity and macroeconomic dynamics". Unpublished Manuscript, Columbia University.

[22] Kim, S., Kollman, R., Kim, J., 2009. "Solving the incomplete markets model with aggregate uncertainty using a perturbation method". *Journal of Economic Dynamics and Control*, doi:10.1016/j.jedc.2008.11.011.

BIOGRAPHICAL SKETCH

Muffasir Badshah received his Bachelors degree in Applied Mathematics from University of Houston-Downtown in 1997. He earned a Dual Masters degree in Mathematics and Computer Science from Drexel University in 2002. He worked on two projects at Drexel - one on short term equity dynamics and endogenous market fluctuations and the other on the properties of a spin market model based on the Ising Hamiltonian. He joined the Department of Statistics at Florida State University where he earned a Masters degree in Mathematical Statistics in 2009 and took interdisciplinary courses in Financial Mathematics to complete his PhD. His strengths are in Computational Statistics. His research interests include Monte Carlo methods with applications to finance, statistical arbitrage methods, and dynamic general equilibrium modeling of heterogeneous agent economies with focus on Asset Pricing.